Generalized inverses in certain Banach algebras of operators*

Lisa A. Oberbroeckling†
Department of Mathematical Sciences
Loyola College in Maryland
4501 N. Charles St.
Baltimore, MD 21210

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Abstract

Let X be a Banach space and T be a bounded linear operator from X to itself ($T \in B(X)$.) An operator $S \in B(X)$ is a generalized inverse of T if $TST = T$. In this paper we look at several Banach algebras of operators and characterize when an operator in that algebra has a generalized inverse that is also in the algebra. Also, Drazin inverses will be related to generalized inverses and spectral projections.

Introduction

Let $B(X)$ denote the space of bounded linear operators from a Banach space X to itself. An operator $T \in B(X)$ has a generalized inverse $S \in B(X)$ if $TST = T$. If X is finite-dimensional ($X = \mathbb{C}^n$), every operator in $B(X) = M_n(\mathbb{C})$ has a generalized inverse. If not, T may or may not have a generalized inverse. Under the conditions where X is infinite-dimensional, the characterization of when an operator $T \in B(X)$ has a generalized inverse in $B(X)$ and methods of the construction of a generalized inverse are well-known [C], [TL].

In Section 1 we look at a Banach algebra called the Jörgens Algebra. This algebra is so named because K. Jörgens presented this algebra in [J] as a way to study integral operators. The algebra and its spectral theory were also studied by B. Barnes in [B1]. In this paper we characterize when an operator in the Jörgens Algebra has a generalized inverse that is also in the algebra. In Section 2 we study Banach spaces that have a bounded inner product. We look at the algebra $\mathcal{B}$ of operators that have an adjoint with respect to this inner product.

*Technical Report 2003-01, Department of Mathematical Sciences, Loyola College in Maryland, 4501 N. Charles St., Baltimore, MD.
†Email: `LOberbroeckling@loyola.edu`

By defining a specific norm on this algebra, it is a Banach *-algebra. We not only study generalized inverses but also Moore-Penrose inverses in this algebra. In Section 3, special conditions on the Jörgens algebra are discussed. The Banach algebras discussed in Section 4 are the commutant and double commutant of an operator $T \in B(X)$. In [K], C. F. King related generalized inverses, the commutant of T and Drazin inverses. We revisit this result and obtain a further result that also involves the double commutant and spectral projections.

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1 The Jörgens Algebra

Let X and Y be Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. Suppose there is a nondegenerate bilinear form $\langle \cdot, \cdot \rangle$ on $X \times Y$ such that for some $M > 0$,

$$|\langle x, y \rangle| \leq M \, \|x\|_X \, \|y\|_Y \text{ for all } x \in X \text{ and } y \in Y. \tag{1.1}$$

Suppose $T \in B(X)$ has an adjoint, denoted $T^\dagger$, with respect to this bilinear form; i.e., $\langle Tx, y \rangle = \langle x, T^\dagger y \rangle$ for all $x \in X$ and $y \in Y$. Define the Jörgens Algebra $\mathcal{A}(X, Y) = \mathcal{A}$ to be

$$\mathcal{A} = \{T \in B(X) \,|\, T^\dagger \text{ exists in } B(Y)\} \text{ with norm } \|T\| = \max\{\|T\|_{op},\ \|T^\dagger\|_{op}\}.$$

With this defined norm, $\mathcal{A}$ is a Banach algebra [J]. $\mathcal{A}$ will denote the Jörgens algebra and we will use the notation $\mathcal{A}(X, Y)$ when it is necessary to specify X and Y. Because the bilinear form is nondegenerate, an operator T in $\mathcal{A}(X, Y)$ is uniquely determined by $T^\dagger$ and vice-versa.

Equation (1.1) gives us continuity of the bilinear form for a fixed $y \in Y$ or a fixed $x \in X$. Thus we can identify $y \in Y$ with an element α_y in the dual space of X (denoted X^*) by $\alpha_y(x) = \langle x, y \rangle$ and likewise we can identify $x \in X$ with an element $\beta_x \in Y^*$. By nondegeneracy of the bilinear form, Y is a total subspace of X^* and X is a total subspace of Y^*. Weak topologies, the $\mathcal{Y}$-topology on X and the $\mathcal{X}$-topology on Y, are formed as in [DS] and these topologies are locally convex. Thus we have for nets $\{x_\delta\} \subseteq X$ and $\{y_\delta\} \subseteq Y$ the following meaning of convergence in these topologies:

$$x_\delta \xrightarrow{\mathcal{Y}} x \text{ means } \langle x_\delta, y \rangle \longrightarrow \langle x, y \rangle \; \forall \, y \in Y;$$

$$y_\delta \xrightarrow{\mathcal{X}} y \text{ means } \langle x, y_\delta \rangle \longrightarrow \langle x, y \rangle \; \forall \, x \in X.$$

Clearly if $Y = X^*$ then the $\mathcal{Y}$-topology is exactly the usual weak topology and the $\mathcal{X}$-topology is the weak*-topology.

Both the $\mathcal{X}$-topology and $\mathcal{Y}$-topology play an important role in studying generalized inverses in the Jörgens algebra. Using Theorem V.3.9 of [DS], we prove the following result pertaining to the Jörgens algebra and the $\mathcal{X}$- and $\mathcal{Y}$-topologies.

Theorem 1.1. *Let $T \in B(X)$. T is $\mathcal{Y}$-continuous if and only if $T \in \mathcal{A}(X,Y)$. Likewise for $S \in B(Y)$, S is $\mathcal{X}$-continuous if and only if $S = T^\dagger$ for some $T \in \mathcal{A}(X,Y)$.*

Proof. First suppose that $T \in \mathcal{A}$ and let $\{x_\delta\}$ be any net in X such that $x_\delta \xrightarrow{\mathcal{Y}} x_o$ for some $x_o \in X$. We then have

$$\langle Tx_\delta, y\rangle = \langle x_\delta, T^\dagger y\rangle \longrightarrow \langle x_o, T^\dagger y\rangle = \langle Tx_o, y\rangle \text{ for all } y \in Y.$$

Thus $Tx_\delta \xrightarrow{\mathcal{Y}} Tx_o$ so T is $\mathcal{Y}$-continuous.

Now suppose that T is $\mathcal{Y}$-continuous. Then for each net $\{x_\delta\} \subseteq X$ such that $x_\delta \xrightarrow{\mathcal{Y}} x_o$ we have $Tx_\delta \xrightarrow{\mathcal{Y}} Tx_o$. In other words, $\langle Tx_\delta, y\rangle \longrightarrow \langle Tx_o, y\rangle$ for each $y \in Y$. Thus the linear functionals on X defined by $\alpha_y(x) := \langle Tx, y\rangle$ for each $y \in Y$ are continuous in the $\mathcal{Y}$-topology. By Theorem V.3.9 of [DS], for each $y \in Y$ there exists a corresponding unique $y' \in Y$ such that $\alpha_y(x) = \langle x, y'\rangle$ for each $x \in X$. Define $T' : Y \longrightarrow Y$ by $T'y := y'$. Clearly T' is well-defined and linear by nondegeneracy and linearity of $\langle \cdot, \cdot \rangle$. Also it is clear that

$$\langle Tx, y\rangle = \langle x, y'\rangle = \langle x, T'y\rangle \text{ for each } x \in X \text{ and } y \in Y.$$

To show $T' \in B(Y)$ it is enough to show that T' is closed by the Closed Graph Theorem. Let $\{y_n\}$ be a sequence in Y, y_o and y elements in Y such that

$$\|y_n - y_o\| \longrightarrow 0 \quad \text{and} \quad \|T'y_n - y\| \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Then we have for any $x \in X$:

$$\begin{aligned} |\langle x, T'y_o - y\rangle| &= |\langle x, T'(y_o - y_n)\rangle + \langle x, T'y_n - y\rangle| \\ &= |\langle Tx, y_o - y_n\rangle + \langle x, T'y_n - y\rangle| \\ &\leq M\,\|T\|_{op}\,\|x\|\,\|y_o - y_n\| + M\,\|x\|\,\|T'y_n - y\| \longrightarrow 0. \end{aligned}$$

Thus $|\langle x, T'y_o - y\rangle| = 0$ for all $x \in X$. By nondegeneracy of the form $T'y_o = y$ so T' is a closed map and so is continuous. Therefore, $T \in \mathcal{A}$ with $T^\dagger = T'$.

Similarly, the result for $S \in B(Y)$ can be shown. $\square$

For subspaces $A \subseteq X$ and $B \subseteq Y$ we have perp-spaces $A^\perp \subseteq Y$ and ${}^\perp B \subseteq X$ defined as

$$\begin{aligned} A^\perp &= \{y \in Y \mid \langle x, y\rangle = 0 \text{ for all } x \in A\} \text{ and} \\ {}^\perp B &= \{x \in X \mid \langle x, y\rangle = 0 \text{ for all } y \in B\}. \end{aligned}$$

It is not hard to show that $A^\perp$ is both norm and $\mathcal{X}$-closed and ${}^\perp B$ is both norm and $\mathcal{Y}$-closed.

Lemma 1.2. *Let M be a subspace of X and N a subspace of Y.*

1. *${}^\perp(M^\perp)$ is the $\mathcal{Y}$-closure of M and $({}^\perp N)^\perp$ is the $\mathcal{X}$-closure of N.*

2. *The subspace M is $\mathcal{Y}$-closed if and only if ${}^{\perp}(M^{\perp}) = M$ and similarly N is $\mathcal{X}$-closed if and only if $({}^{\perp}N)^{\perp} = N$.*

3. *For any $T \in \mathcal{A}$, $\mathcal{N}(T)$ is $\mathcal{Y}$-closed and $\mathcal{N}(T^{\dagger})$ is $\mathcal{X}$-closed.*

4. *For any $T \in \mathcal{A}$, $\mathcal{R}(T^{\dagger}) \subseteq \mathcal{N}(T)^{\perp}$ and $\mathcal{R}(T) \subseteq {}^{\perp}\mathcal{N}(T^{\dagger})$.*

The first two results are direct corollaries of the Hahn-Banach theorem while the third follows from the Hahn-Banach theorem and Theorem 1.1. The fourth result is clear.

For an operator $T \in \mathcal{A}$ one can consider when $\mathcal{R}(T)^{\perp} = \mathcal{N}(T^{\dagger})$, $\mathcal{N}(T)^{\perp} = \mathcal{R}(T^{\dagger})$, ${}^{\perp}\mathcal{R}(T^{\dagger}) = \mathcal{N}(T)$ and ${}^{\perp}\mathcal{N}(T^{\dagger}) = \mathcal{R}(T)$.

Lemma 1.3. *Let $T \in \mathcal{A}$.*

1. *$\mathcal{R}(T)^{\perp} = \mathcal{N}(T^{\dagger})$;*

2. *${}^{\perp}\mathcal{R}(T^{\dagger}) = \mathcal{N}(T)$;*

3. *${}^{\perp}\mathcal{N}(T^{\dagger}) = \mathcal{R}(T)$ exactly when $\mathcal{R}(T)$ is $\mathcal{Y}$-closed and*

4. *$\mathcal{N}(T)^{\perp} = \mathcal{R}(T^{\dagger})$ exactly when $\mathcal{R}(T^{\dagger})$ is $\mathcal{X}$-closed.*

Proof. Clearly $\mathcal{N}(T^{\dagger}) \subseteq \mathcal{R}(T)^{\perp}$ and $\mathcal{N}(T) \subseteq {}^{\perp}\mathcal{R}(T^{\dagger})$. Let $y \in \mathcal{R}(T)^{\perp}$ be arbitrary. Then
$$\langle x, T^{\dagger}y\rangle = \langle Tx, y\rangle = 0 \quad \text{for all } x \in X.$$
By nondegeneracy of the form, $T^{\dagger}y = 0$ so $y \in \mathcal{N}(T^{\dagger})$, thus $\mathcal{R}(T)^{\perp} = \mathcal{N}(T^{\dagger})$. By a similar argument, ${}^{\perp}\mathcal{R}(T^{\dagger}) = \mathcal{N}(T)$. From these two equalities we get ${}^{\perp}(\mathcal{R}(T)^{\perp}) = {}^{\perp}\mathcal{N}(T^{\dagger})$ and $({}^{\perp}\mathcal{R}(T^{\dagger}))^{\perp} = \mathcal{N}(T)^{\perp}$. From Lemma 1.2 we obtain the last two results of the lemma. □

We now have the following useful lemma.

Lemma 1.4. *The following are true for any projection $P \in \mathcal{A}$:*

1. *$\mathcal{N}(P) = {}^{\perp}\mathcal{R}(P^{\dagger})$;*

2. *$\mathcal{R}(P) = {}^{\perp}\mathcal{N}(P^{\dagger})$;*

3. *$\mathcal{R}(P^{\dagger}) = \mathcal{N}(P)^{\perp}$; and*

4. *$\mathcal{N}(P^{\dagger}) = \mathcal{R}(P)^{\perp}$.*

Thus $\mathcal{R}(P)$ and $\mathcal{N}(P)$ are both $\mathcal{Y}$-closed and $\mathcal{R}(P^{\dagger})$ and $\mathcal{N}(P^{\dagger})$ are both $\mathcal{X}$-closed.

Proof. To prove the first two notice that both P and $I - P$ are in $\mathcal{A}$. From Lemma 1.2, both $\mathcal{N}(P)$ and $\mathcal{N}(I - P) = \mathcal{R}(P)$ are $\mathcal{Y}$-closed and thus Lemma 1.3 applies. The last two equalities use the same argument on $P^{\dagger}$ and $I - P^{\dagger}$. □

We immediately have the following theorem.

Theorem 1.5. *Let P be a projection in $B(X)$. Then $Y = \mathcal{R}(P)^\perp \oplus \mathcal{N}(P)^\perp$ if and only if $P \in \mathcal{A}$.*

Proof. First assume that $Y = \mathcal{R}(P)^\perp \oplus \mathcal{N}(P)^\perp$. Then for any $x \in X$ and $y \in Y$ we have unique representations

$$\begin{array}{rcll} x & = & x_1 + x_2, & x_1 \in \mathcal{R}(P),\ x_2 \in \mathcal{N}(P) \text{ and} \\ y & = & y_1 + y_2, & y_1 \in \mathcal{R}(P)^\perp,\ y_2 \in \mathcal{N}(P)^\perp. \end{array}$$

Note that $\langle x_1, y_1\rangle = 0$ and $\langle x_2, y_2\rangle = 0$. Since $\mathcal{N}(P)^\perp$ and $\mathcal{R}(P)^\perp$ are both norm-closed subspaces, we can define $Q \in B(Y)$ to be the continuous projection onto $\mathcal{N}(P)^\perp$ with nullspace $\mathcal{R}(P)^\perp$ [TL, Theorem IV.12.2]. Then for any $x \in X$ and $y \in Y$ and the above representations,

$$\begin{aligned} \langle x, Qy\rangle &= \langle x, y_2\rangle \\ &= \langle x_1, y_2\rangle + \langle x_2, y_2\rangle \\ &= \langle x_1, y_2\rangle \\ &= \langle x_1, y_1\rangle + \langle x_1, y_2\rangle \\ &= \langle Px, y\rangle. \end{aligned}$$

So $\langle Px, y\rangle = \langle x, Qy\rangle$ for all $x \in X$, $y \in Y$. Thus $P \in \mathcal{A}$ with $P^\dagger = Q$.

Now assume $P \in \mathcal{A}$. Clearly $P^\dagger \in B(Y)$ is a projection so $Y = \mathcal{N}(P^\dagger) \oplus \mathcal{R}(P^\dagger)$. By the above lemma, $\mathcal{N}(P^\dagger) = \mathcal{R}(P)^\perp$. Also, if we let $Q = I - P$, $Q \in \mathcal{A}$ so $\mathcal{R}(P^\dagger) = \mathcal{N}(Q^\dagger) = \mathcal{R}(Q)^\perp = \mathcal{N}(P)^\perp$. Thus $Y = \mathcal{R}(P)^\perp \oplus \mathcal{N}(P)^\perp$. $\square$

Before discussing generalized inverses in $\mathcal{A}$ one should first discuss invertibility and Fredholm theory in $\mathcal{A}$. If an operator $T \in \mathcal{A}$ is invertible in $\mathcal{A}$, it is clear that $(T^{-1})^\dagger$ must equal $(T^\dagger)^{-1}$. Thus $T^\dagger$ must also be invertible in $B(Y)$. We will denote the Fredholm operators in $B(X)$ by $\Phi(X)$, the index of T by $\iota(T)$ and the Fredholm operators of index zero by $\Phi^0(X)$.

Definition 1.6. Let $\Phi_{\mathcal{A}}$ be the set of all operators in $\mathcal{A}$ that are invertible modulo the set of finite rank operators in $\mathcal{A}$; i.e., $T \in \Phi_{\mathcal{A}}$ if there exist an $S \in \mathcal{A}$ and finite rank operators K, $J \in \mathcal{A}$ such that $TS = I - K$ and $ST = I - J$.

This definition was discussed in [B1] and shown to be a natural definition. Let $\Phi^0_{\mathcal{A}}$ denote the set of all operators in $\Phi_{\mathcal{A}}$ having index zero. To consider when an arbitrary operator in $\mathcal{A}$ has a generalized inverse in $\mathcal{A}$ we must consider the different topologies on X and Y and how the nullspaces and ranges of T and $T^\dagger$ are related. The following result characterizes the existence of generalized inverses in the Jörgens algebra.

Theorem 1.7. *Let $T \in \mathcal{A}(X, Y)$. T has a generalized inverse $S \in \mathcal{A}(X, Y)$ if and only if*

1. *There exist projections P and Q in $\mathcal{A}(X, Y)$ such that*

$$\mathcal{R}(P) = \mathcal{N}(T), \quad \mathcal{R}(Q) = \mathcal{R}(T); \quad \textit{and}$$

2. $\mathcal{R}(T^\dagger) = \mathcal{N}(T)^\perp$.

Proof. First assume that there exists a generalized inverse S of T such that $S \in \mathcal{A}$. Then $S^\dagger$ is a generalized inverse of $T^\dagger$. By Theorem IV.12.9 of [TL], there exist continuous projections $P = I - ST$ and $Q = TS$ such that $\mathcal{R}(P) = \mathcal{N}(T)$ and $\mathcal{R}(Q) = \mathcal{R}(T)$. Clearly by construction P and Q are in $\mathcal{A}$ with $P^\dagger = I - T^\dagger S^\dagger$ and $Q^\dagger = S^\dagger T^\dagger$. By that same theorem, $T^\dagger S^\dagger$ is a projection onto $\mathcal{R}(T^\dagger)$; thus $\mathcal{N}(P^\dagger) = \mathcal{R}(T^\dagger)$. However, by Lemma 1.4, $\mathcal{N}(P^\dagger) = \mathcal{R}(P)^\perp = \mathcal{N}(T)^\perp$. Therefore, $\mathcal{R}(T^\dagger) = \mathcal{N}(T)^\perp$.

Conversely, suppose $\mathcal{R}(T^\dagger) = \mathcal{N}(T)^\perp$ and let P and Q be the projections as in the hypothesis (1). Let $L = \mathcal{N}(P)$ and $M = \mathcal{N}(Q)$. Then

$$X = \mathcal{N}(T) \oplus L = \mathcal{R}(T) \oplus M$$

and from Theorem 1.5,

$$Y = \mathcal{N}(T)^\perp \oplus L^\perp = \mathcal{R}(T)^\perp \oplus M^\perp.$$

By Lemma 1.4,

$$\begin{aligned} \mathcal{N}(P^\dagger) &= \mathcal{R}(P)^\perp = \mathcal{N}(T)^\perp = \mathcal{R}(T^\dagger), \qquad & \mathcal{R}(P^\dagger) &= L^\perp, \\ \mathcal{N}(Q^\dagger) &= \mathcal{R}(Q)^\perp = \mathcal{R}(T)^\perp = \mathcal{N}(T^\dagger), \qquad & \mathcal{R}(Q^\dagger) &= M^\perp. \end{aligned}$$

Define the map $T_1 : L \longrightarrow \mathcal{R}(T)$ by $T_1 x = Tx$ for all $x \in L$. Clearly T_1 is a linear, bounded, one-to-one operator onto $\mathcal{R}(T)$. Since $\mathcal{R}(T)$ is norm-closed, it is a Banach space. Thus by the Open Mapping Theorem, $T_1^{-1} : \mathcal{R}(T) \longrightarrow L$ exists as a bounded linear operator. Define $S : X \longrightarrow X$ to be $T_1^{-1}Q$. Clearly $S \in B(X)$ and $STx = x$ for all $x \in L$. Let $x \in X$ be arbitrary. Since x can be expressed uniquely as $x = x_1 + x_2$ with $x_1 \in \mathcal{N}(T)$ and $x_2 \in L$,

$$\begin{aligned} TSTx &= TST(x_1 + x_2) \\ &= TSTx_2 \\ &= Tx_2 \\ &= T(x_1 + x_2) \\ &= Tx. \end{aligned}$$

Thus S is a generalized inverse for T.

Define $T_2 : M^\perp \longrightarrow \mathcal{R}(T^\dagger)$ by $T_2 y = T^\dagger y$. Clearly T_2 is a bounded linear operator. It is one-to-one and onto $\mathcal{R}(T^\dagger)$ since $\mathcal{R}(T)^\perp = \mathcal{N}(T^\dagger)$. Since $\mathcal{R}(T^\dagger)$ is $\mathcal{X}$-closed, it is norm-closed so $T_2^{-1} : \mathcal{R}(T^\dagger) \longrightarrow M^\perp$ exists as a bounded linear operator by the Open Mapping Theorem. Define $S_2 \in B(Y)$ by $S_2 = T_2^{-1}(I - P^\dagger)$. For any $y \in Y$ we have the unique representation $y = y_1 + y_2$ with $y_1 \in \mathcal{N}(T^\dagger) = \mathcal{R}(T)^\perp$ and $y_2 \in M^\perp$. Note that

$$T^\dagger y_2 = T_2 y_2 \quad \text{and} \quad S_2 T^\dagger y_2 = T_2^{-1} T^\dagger y_2 = T_2^{-1} T_2 y_2 = y_2.$$

Then we have for any $y \in Y$ with $y = y_1 + y_2$ as above,

$$\begin{aligned} T^\dagger S_2 T^\dagger y &= T^\dagger S_2 T^\dagger (y_1 + y_2) \\ &= T^\dagger S_2 T^\dagger y_2 \\ &= T^\dagger y_2 \\ &= T^\dagger (y_1 + y_2) \\ &= T^\dagger y. \end{aligned}$$

Thus S_2 is a generalized inverse of $T^\dagger$. For any $x \in X$ and $y \in Y$ we have

$$\begin{aligned} x =& Tx_1 + x_2, \qquad x_1 \in L, \quad x_2 \in M \text{ and} \\ y =& T^\dagger y_1 + y_2, \qquad y_1 \in M^\perp, \quad y_2 \in L^\perp. \end{aligned}$$

By Lemma 1.4, $\mathcal{N}(I - P^\dagger) = \mathcal{R}(P^\dagger) = L^\perp$ so $S_2 y_2 = T_2^{-1}(I - P^\dagger) y_2 = 0$. Thus

$$S_2 y = S_2 T^\dagger y_1 + S_2 y_2 = S_2 T^\dagger y_1 = y_1 \text{ since } y_1 \in M^\perp$$

and

$$\begin{aligned} \langle Sx, y\rangle &= \langle STx_1, y\rangle + \langle Sx_2, y\rangle \\ &= \langle x_1, T^\dagger y_1\rangle \qquad (Sx_2 = 0 \text{ since } S = T_1^{-1}Q = 0 \text{ on } \mathcal{N}(Q) = M) \\ &= \langle Tx_1, y_1\rangle \\ &= \langle Tx_1 + x_2, y_1\rangle \qquad (\text{since } x_2 \in M, y_1 \in M^\perp) \\ &= \langle x, S_2 y\rangle. \end{aligned}$$

Therefore S is a generalized inverse of T in $\mathcal{A}$ with $S^\dagger = S_2$. □

Remark 1.8. With the above construction of $S = T_1^{-1}Q$, $STS = S$. This is because $TT_1^{-1}x = x$ for all $x \in R(T)$ and $QT = T$ since $\mathcal{R}(Q) = \mathcal{R}(T)$ so we have

$$\begin{aligned} STS &= T_1^{-1}QTT_1^{-1}Q \\ &= T_1^{-1}TT_1^{-1}Q \\ &= T_1^{-1}Q \\ &= S. \end{aligned}$$

From the theorem and Lemma 1.4 we obtain the following corollary.

Corollary 1.9. *Let $T \in \mathcal{A}$ such that T has a generalized inverse $S \in \mathcal{A}$. Then $\mathcal{R}(T) = {}^\perp\mathcal{N}(T^\dagger)$ and $\mathcal{R}(T^\dagger) = \mathcal{N}(T)^\perp$.*

2 Banach Spaces with Bounded Inner Product

Let X be a Banach space with a bounded inner product $(\cdot,\cdot)$. For $T \in B(X)$, define T^* to be the adjoint of T with respect to the inner product. That is,

$$(Tx, y) = (x, T^*y) \text{ for all } x, y \in X.$$

Define the algebra $\mathcal{B} = \{T \in B(X) \big| \exists T^* \in B(X)\}$. This is equivalent to the algebra of all bounded linear operators on X that have bounded extensions to the Hilbert space completion of X [L]. Define a norm on the elements of $\mathcal{B}$ similar to the Jörgens algebra; that is, for $T \in \mathcal{B}$

$$\|T\| = \max\{\|T\|_{op}, \|T^*\|_{op}\}.$$

This makes $\mathcal{B}$ a Banach *-algebra and so Moore-Penrose inverses can be discussed. If $\mathcal{B}$ is a *-algebra, $b \in \mathcal{B}$ is a Moore-Penrose inverse of $a \in \mathcal{B}$ if

$$aba = a, \quad bab = b, \quad (ba)^* = ba \quad \text{and} \quad (ab)^* = ab.$$

Throughout the rest of this section, $\mathcal{B}$ will denote the *-algebra above with the inner product space X and T^* will denote the adjoint in this algebra. As in the Jörgens algebra case we can define the space $M^\perp \subseteq X$ for a subspace M of X. For a fixed $x_o \in X$ define $\alpha_{x_o}(x) := (x, x_o)$. This is clearly a linear functional and by continuity of the inner product, $\alpha_{x_o} \in X^*$. Thus we have a weak $\mathcal{X}$-topology on X as defined in [DS] and the Jörgens algebra case. All of the results about the $M^\perp$ spaces and the $\mathcal{X}$-topology in the Jörgens algebra case apply. In particular we have the following results.

Lemma 2.1. *The following are true for any projection $P \in \mathcal{B}$.*

1. $\mathcal{N}(P) = \mathcal{R}(P^*)^\perp$*;*
2. $\mathcal{R}(P) = \mathcal{N}(P^*)^\perp$*;*
3. $\mathcal{R}(P^*) = \mathcal{N}(P)^\perp$*;*
4. $\mathcal{N}(P^*) = \mathcal{R}(P)^\perp$*.*

Thus $\mathcal{R}(P)$, $\mathcal{N}(P)$, $\mathcal{R}(P^*)$ and $\mathcal{N}(P^*)$ are all $\mathcal{X}$-closed.

Theorem 2.2. *Let P be a projection in $B(X)$. $P \in \mathcal{B}$ if and only if $X = \mathcal{R}(P)^\perp \oplus \mathcal{N}(P)^\perp$.*

Theorem 2.3. *Let $T \in \mathcal{B}$. T has a generalized inverse in $\mathcal{B}$ if and only if*

1. *There exist projections P and Q in $\mathcal{B}$ with*

$$\mathcal{R}(P) = \mathcal{N}(T) \quad \textit{and} \quad \mathcal{R}(Q) = \mathcal{R}(T); \quad \textit{and}$$

2. $\mathcal{R}(T^*) = \mathcal{N}(T)^\perp$*.*

The proofs of these theorems are the same as the Jörgens algebra case since the only difference is that there is a sesquilinear form rather than a bilinear form.

We immediately have the following result.

Theorem 2.4. *Let $T \in \mathcal{B}$. T has a Moore-Penrose inverse in $\mathcal{B}$ if and only if*

1. *$X = \mathcal{N}(T) \oplus \mathcal{N}(T)^{\perp} = \mathcal{R}(T) \oplus \mathcal{R}(T)^{\perp}$; and*
2. *$\mathcal{R}(T) = \mathcal{N}(T^*)^{\perp}$ and $\mathcal{R}(T^*) = \mathcal{N}(T)^{\perp}$;*

Proof. First assume that T has a Moore-Penrose inverse $S \in \mathcal{B}$. By definition, S is a generalized inverse of T in $\mathcal{B}$ and there exist selfadjoint projections $P = I - ST$ and $Q = TS$ in $\mathcal{B}$ such that

$$\mathcal{R}(P) = \mathcal{N}(T) \quad \text{and} \quad \mathcal{R}(Q) = \mathcal{R}(T).$$

From Lemma 2.1,

$$\mathcal{N}(P) = \mathcal{N}(P^*) = \mathcal{R}(P)^{\perp} = \mathcal{N}(T)^{\perp} \text{ and } \mathcal{N}(Q) = \mathcal{N}(Q^*) = \mathcal{R}(Q)^{\perp} = \mathcal{R}(T)^{\perp}.$$

Thus $X = \mathcal{N}(T)\oplus\mathcal{N}(T)^{\perp} = \mathcal{R}(T)\oplus\mathcal{R}(T)^{\perp}$. By Theorem 2.3, $\mathcal{R}(T^*) = \mathcal{N}(T)^{\perp}$. By Lemma 2.1, $\mathcal{R}(Q) = \mathcal{R}(T)$ is $\mathcal{X}$-closed; thus $\mathcal{R}(T) = \mathcal{N}(T^*)^{\perp}$.

Now assume the converse. Let P be the projection onto $\mathcal{N}(T)$ along $\mathcal{N}(T)^{\perp}$ and Q be the projection onto $\mathcal{R}(T)$ along $\mathcal{R}(T)^{\perp}$. Clearly by Theorem 2.2 both P and Q are in $\mathcal{B}$ and from Lemma 2.1 we have

$$\begin{aligned} \mathcal{R}(P^*) &= \mathcal{N}(P)^{\perp} = \mathcal{N}(T), \qquad & \mathcal{N}(P^*) &= \mathcal{R}(P)^{\perp} = \mathcal{N}(T)^{\perp} \\ \mathcal{R}(Q^*) &= \mathcal{N}(Q)^{\perp} = \mathcal{R}(T), \qquad & \mathcal{N}(Q^*) &= \mathcal{R}(Q)^{\perp} = \mathcal{R}(T)^{\perp}. \end{aligned}$$

Thus $P^* = P$ and $Q^* = Q$. By Theorem 2.3, T has a generalized inverse $S \in \mathcal{B}$ such that $P = I - ST$, $Q = TS$ and $STS = S$. But $I - (ST)^* = P^* = P = I - ST$; thus $ST = (ST)^*$. Also $(TS)^* = Q^* = Q = TS$. Thus $S \in \mathcal{B}$ is a Moore-Penrose inverse of T. □

As in the Jörgens algebra case, an operator T is invertible in $\mathcal{B}$ if and only if T and T^* are invertible in $B(X)$ [B1, Theorem 2.5]. Also, we say T is Fredholm with respect to $\mathcal{B}$, or $T \in \Phi_{\mathcal{B}}$, when T is invertible modulo finite rank operators in $\mathcal{B}$; i.e., there exists an operator $S \in \mathcal{B}$ and finite rank operators $F, G \in \mathcal{B}$ such that $ST = I + F$ and $TS = I + G$. Let $T \in \Phi^0_{\mathcal{B}}$ denote the set of operators in $\Phi_{\mathcal{B}}$ of index zero. Also, $T \in \Phi_{\mathcal{B}}$ if and only if $T \in \Phi(X)$, $T^* \in \Phi(X)$ and $\iota(T) + \iota(T^*) = 0$ [B1].

Elements of C*-algebras that have generalized inverses also have Moore-Penrose inverses [HM, Theorem 6]. The proof of this result uses the symmetric property that for any element x of a C*-algebra, $I + x^*x$ is invertible. In $\mathcal{B}$ we do not necessarily have symmetry so we first need some preliminaries.

Lemma 2.5. *Let P be a projection in $\mathcal{B}$. Then $\mathcal{N}(P) = \mathcal{N}(PP^*P)$ and $\mathcal{N}(P^*) = \mathcal{N}(P^*PP^*)$.*

Proof. We only need to prove the first equality. Clearly $\mathcal{N}(P) \subseteq \mathcal{N}(PP^*P)$. To prove the reverse inclusion we use the fact that for any subspace $M \subseteq X$, $M \cap M^\perp = \{0\}$. Let $PP^*Px = 0$. By Lemma 2.1, $P^*Px \in \mathcal{N}(P) \cap R(P^*) = \mathcal{N}(P) \cap \mathcal{N}(P)^\perp = \{0\}$ and therefore $Px \in \mathcal{N}(P^*) \cap \mathcal{R}(P) = \mathcal{R}(P)^\perp \cap \mathcal{R}(P) = \{0\}$. Thus $x \in \mathcal{N}(P)$ and we have equality. $\square$

Note that the above lemma is true for any $T \in \mathcal{B}$ such that $\mathcal{R}(T^*) = \mathcal{N}(T)^\perp$ and $\mathcal{R}(T) = \mathcal{N}(T^*)^\perp$ (the other two equalities are true for any $T \in \mathcal{B}$).

Lemma 2.6. *Let P be a projection in $\mathcal{B}$ and $U = I + (P - P^*)^*(P - P^*) = I - (P - P^*)^2$. Then U is injective.*

Proof. Suppose $Ux = 0$ for some $x \in X$. By definition of U,

$$x = Px + P^*x - PP^*x - P^*Px.$$

By multiplying the equation by P and P^* separately, we get both $PP^*Px = 0$ and $P^*PP^*x = 0$. Therefore $x \in \mathcal{N}(PP^*P) \cap \mathcal{N}(P^*PP^*) = N(P) \cap \mathcal{N}(P^*)$ by Lemma 2.5. Consequently $\mathcal{N}(U) \subseteq \mathcal{N}(P) \cap \mathcal{N}(P^*)$.

Clearly $U \in \mathcal{B}$ and $U^* = U$. So we have $\mathcal{R}(U) = \mathcal{R}(U^*) \subseteq \mathcal{N}(U)^\perp$ and $\mathcal{N}(U) = \mathcal{R}(U)^\perp$. Let $y \in \mathcal{N}(U)$. Then for all $x \in X$,

$$\begin{aligned}
0 &= (x, Uy) \\
&= (Ux, y) \\
&= (x - Px - P^*x + PP^*x + P^*Px, y) \\
&= (x, y) - (Px, y) - (P^*x, y) + (PP^*x, y) + (P^*Px, y) \\
&= (x, y) - (x, P^*y) - (x, Py) + (P^*x, P^*y) + (Px, Py) \\
&= (x, y)
\end{aligned}$$

since $y \in \mathcal{N}(P) \cap \mathcal{N}(P^*)$. So for any $y \in \mathcal{N}(U)$, $(x, y) = (Ux, y) = 0$ for all $x \in X$. By nondegeneracy of the inner product, $y = 0$ and so U in injective. $\square$

Theorem 2.7. *Let $T \in \mathcal{B}$ such that T has a generalized inverse $S \in \mathcal{B}$. Let $P = ST$ and $Q = TS$. If $U = I - (P - P^*)^2$ and $V = I - (Q - Q^*)^2$ are both surjective then T has a Moore-Penrose inverse $\widehat{S}$ in $\mathcal{B}$ defined by $\widehat{S} = P^*PU^{-1}SQQ^*V^{-1}$.*

Proof. Let $S \in \mathcal{B}$ be the generalized inverse of $T \in \mathcal{B}$ and let $P = ST$ and $Q = TS$. Let $U = I + (P - P^*)^*(P - P^*) = I - (P - P^*)^2$ and $V = I + (Q - Q^*)^*(Q - Q^*) = I - (Q - Q^*)^2$. Clearly U and V are in $\mathcal{B}$ and both self-adjoint. By the above lemma both U and V are injective so U and V are both invertible in $B(X)$. However, $U = U^*$ and $V = V^*$, thus U and V are both invertible in $\mathcal{B}$.

Now we apply the proof of [HM, Theorem 6]. The theorem states that if an element in a C*-algebra has a generalized inverse then it has a Moore-Penrose inverse in the algebra. In fact, the proof works in any *-algebra where U and V as defined above are invertible and the proof then follows. It should be noted that the Moore-Penrose inverse $\widehat{S}$ of T is constructed in [HM, Theorem 6] as follows:

$$\widehat{S} = P^*PU^{-1}SQQ^*V^{-1}$$

where S is the generalized inverse of T in $\mathcal{B}$, $P = ST$, $Q = TS$, $U = I + (P - P^*)^*(P - P^*)$ and $V = I + (Q - Q^*)^*(Q - Q^*)$. $\square$

Corollary 2.8. *Let $T \in \mathcal{B}$ such that $T \in \Phi_{\mathcal{B}}$. Then T has a Moore-Penrose inverse in $\mathcal{B}$.*

Proof. Since $T \in \Phi_{\mathcal{B}}$, T has a generalized inverse $S \in \Phi_{\mathcal{B}}$ [J, Theorem 5.16]. Clearly T^* and S^* are both in $\Phi_{\mathcal{B}}$. The projections $P = ST$, $Q = TS$, P^* and Q^* are all in $\Phi^0_{\mathcal{B}}$ [TL, Theorem IV.13.1] with $\mathcal{N}(P) = \mathcal{N}(T)$ and $\mathcal{R}(Q) = \mathcal{R}(T)$.

Let $U = I + (P - P^*)^*(P - P^*) = I - (P - P^*)^2$ and $V = I + (Q - Q^*)^*(Q - Q^*) = I - (Q - Q^*)^2$. As above, U and V are in $\mathcal{B}$ and both self-adjoint. Clearly $PP^* \in \Phi^0_{\mathcal{B}}$. Note that $I - P$ is of finite rank since $\mathcal{R}(I - P) = \mathcal{N}(P) = \mathcal{N}(T)$. By [TL, Theorem IV.13.4] we then have $P^*P - (I - P)P^* \in \Phi^0_{\mathcal{B}}$ since $(I - P)P^*$ is of finite rank. Using the same theorem shows that $U = I - P + (P^*P - (I - P)P^*) \in \Phi^0_{\mathcal{B}}$. A similar argument on Q shows that $V \in \Phi^0_{\mathcal{B}}$.

By Lemma 2.6 both U and V are injective and since both are of index zero the operators are also surjective. Thus we apply the previous theorem to get the Moore-Penrose inverse $\widehat{S}$ of T defined by $\widehat{S} = P^*PU^{-1}SQQ^*V^{-1} \in \mathcal{B}$. $\square$

3 Extension Algebras

For any operator $T \in B(X)$ such that $I - T$ has generalized inverse $\widehat{W} \in B(X)$, we can write $\widehat{W} = I - W$, where $W = I - \widehat{W}$. Thus one can always assume any generalized inverse of $I - T$ is of the form $I - W$ where $W \in B(X)$.

Lemma 3.1. *Let $T \in B(X)$ be such that $I - T$ has a generalized inverse $I - W$ where $W \in B(X)$. Then $I - (W - I)T$ is also a generalized inverse of $I - T$.*

Proof. The proof is purely computational.

$$\begin{aligned} I - (W - I)T &= I + (I - W)T, \quad \text{thus} \\ (I - T)[I - (W - I)T](I - T) &= (I - T)[I + (I - W)T](I - T) \\ &= [(I - T) + (I - T)(I - W)T](I - T) \\ &= (I - T)^2 + (I - T)(I - W)T(I - T) \\ &= (I - T)^2 + (I - T)(I - W)(I - T)T \\ &= (I - T)^2 + (I - T)T \\ &= I - T. \end{aligned}$$

$\square$

Let $T \in \mathcal{A}(X, Y)$ be such that $\mathcal{R}(T^*) \subseteq Y$. Define $R : Y \longrightarrow X^*$ by the inclusion map: $y \mapsto \alpha_y$ where $\alpha_y(x) = \langle x, y\rangle$. Since $|\langle x, y\rangle| \leq M\|x\|_X \|y\|_Y$,

$$\begin{aligned}\|Ry\| = \|\alpha_y\| &= \sup\{|\alpha_y(x)| : \|x\|_X \leq 1\} \\ &= \sup\{|\langle x, y\rangle| : \|x\|_X \leq 1\} \\ &\leq M\|y\|_Y.\end{aligned}$$

Thus R is a bounded linear operator. Since $\mathcal{R}(T^*) \subseteq Y$, we can define $S : X^* \longrightarrow Y$ by $S\alpha = T^*\alpha$. Clearly S is also a bounded linear operator and we have

$$RS = T^* \qquad SR = T^\dagger.$$

The results of B. Barnes in [B2] then apply to $I - RS = I - T^*$ and $I - SR = I - T^\dagger$. In particular, we have the following results.

Theorem 3.2. *Let $T \in \mathcal{A}(X, Y)$ be such that $\mathcal{R}(T^*) \subseteq Y$. Then $I - T \in \Phi_{\mathcal{A}}$ if and only if $I - T \in \Phi(X)$.*

Proof. Recall that $I - T \in \Phi_{\mathcal{A}}$ if and only if the following three things occur: $I - T \in \Phi(X)$, $I - T^\dagger \in \Phi(Y)$ and $\iota(I - T) + \iota(I - T^\dagger) = 0$ [B1, Theorem 2.5]. If $I - T \in \Phi(X)$ then $I - T^* \in \Phi(X^*)$ and $\iota(I - T) + \iota(I - T^*) = 0$. By [B2, Theorem 6], $I - T^\dagger \in \Phi(Y)$ and $\iota(I - T^\dagger) = \iota(I - T^*)$. So $I - T \in \Phi_{\mathcal{A}}$. □

If T satisfies the above conditions and $I - T \in \Phi(X)$, $I - T$ has a generalized inverse in $\mathcal{A}(X, Y)$. But more can be said in general.

Theorem 3.3. *Let $T \in \mathcal{A}(X, Y)$ be such that $\mathcal{R}(T^*) \subseteq Y$. The operator $I - T$ has a generalized inverse in $\mathcal{A}$ if and only if $I - T$ has a generalized inverse in $B(X)$.*

Proof. If $W \in B(X)$ such that $I - W$ is a generalized inverse of $I - T$ then $I - W^*$ is a generalized inverse of $I - T^*$. In particular, $I - T^*$ has a generalized inverse if and only if $I - T^\dagger$ has a generalized inverse [B2, Theorem 4]. If $I - W^*$ is the generalized inverse of $I - RS = I - T^*$ then through the proof one can build the generalized inverse $I - V$ of $I - SR = I - T^\dagger$, where $V = S(W^* - I)R$.

By definition of $R : Y \longrightarrow X^*$, $\langle x, y\rangle = (Ry)(x)$ for any $x \in X$ and $y \in Y$. Also, recall that $RS = T^*$. Let $x \in X$ and $y \in Y$ be arbitrary. We then have

$$\begin{aligned}\langle x, Vy\rangle &= \langle x, S(W^* - I)Ry\rangle \\ &= [RS(W^* - I)Ry](x) \\ &= [T^*(W^* - I)Ry](x) \\ &= (Ry)[(W - I)Tx] \\ &= \langle (W - I)Tx, y\rangle.\end{aligned}$$

Thus $(W - I)T \in \mathcal{A}$ with $[(W - I)T]^\dagger = V = S(W^* - I)R$. From Lemma 3.1, $I - (W - I)T$ is a generalized inverse of $I - T$. Thus $I - (W - I)T \in \mathcal{A}$ with $[I - (W - I)T]^\dagger = I - V$ and so $I - T$ has a generalized inverse in $\mathcal{A}$. □

Now consider the situation in which X and Y are Banach spaces with X dense in Y and there is a continuous embedding $J : X \hookrightarrow Y$, $Jx = x$ for all $x \in X$. In this situation one can form a Jörgens algebra to obtain some results concerning extensions of bounded linear operators from X to Y.

Define a bilinear form $\langle \cdot, \cdot \rangle$ on $X \times Y^*$ by

$$\langle x, \alpha \rangle = \alpha(Jx) \text{ for } x \in X,\ \alpha \in Y^*.$$

Since X is dense in Y, the form is nondegenerate and we have the inequality

$$|\langle x, \alpha \rangle| = |\alpha(Jx)| \leq \|\alpha\|_{Y^*} \|J\|_{op} \|x\|_X. \tag{3.1}$$

Let $\mathcal{E} = \{T \in B(X) |\ \exists \text{ continuous extension } T_e \in B(Y) \text{ of } T\}$. Note that for $T \in \mathcal{E}$, $x \in X$ and $\alpha \in Y^*$ we have the following:

$$\begin{aligned}
\langle Tx, \alpha \rangle &= \alpha(JTx) \\
&= (\alpha \circ T_e)(Jx) \\
&= (T_e^* \alpha)(Jx) \\
&= \langle x, (T_e)^* \alpha \rangle.
\end{aligned}$$

Suppose $T \in B(X)$ is an operator that has an adjoint $T^\dagger$ relative to this bilinear form; i.e., $\langle x, T^\dagger \alpha \rangle = \langle Tx, \alpha \rangle$ for all $x \in X$ and $\alpha \in Y^*$. Clearly $T^\dagger : Y^* \longrightarrow Y^*$ is linear. Now suppose that $\{\alpha_n\} \subseteq Y^*$ is a sequence and there exist elements α and $\alpha_o \in Y^*$ such that

$$\|\alpha_n - \alpha\|_{Y^*} \longrightarrow 0 \quad \text{and} \quad \|T^\dagger \alpha_n - \alpha_o\|_{Y^*} \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

From inequality (3.1),

$$\begin{aligned}
|\langle x, T^\dagger \alpha - \alpha_o \rangle| &= |\langle x, T^\dagger \alpha \rangle - \langle x, T^\dagger \alpha_n \rangle + \langle x, T^\dagger \alpha_n \rangle - \langle x, \alpha_o \rangle| \\
&= |\langle x, T^\dagger (\alpha - \alpha_n) \rangle + \langle x, T^\dagger \alpha_n - \alpha_o \rangle| \\
&= |\langle Tx, \alpha - \alpha_n \rangle + \langle x, T^\dagger \alpha_n - \alpha_o \rangle| \\
&\leq \|\alpha - \alpha_n\|_{Y^*} \|J\|_{op} \|Tx\|_X + \|x\|_X \|J\|_{op} \|T^\dagger \alpha_n - \alpha_o\|_{Y^*} \\
&\longrightarrow 0.
\end{aligned}$$

Thus $\langle x, T^\dagger \alpha - \alpha_o \rangle = 0$ for all $x \in X$. By nondegeneracy of the form, $T^\dagger \alpha = \alpha_o$ and so $T^\dagger$ is a closed operator. By the Closed Graph Theorem, $T^\dagger \in B(Y^*)$ and we have $T^\dagger \alpha$ restricted to X is equal to $\alpha \circ T$. Thus $\mathcal{E} = \mathcal{A}(X, Y^*)$ with the above bilinear form and $T^\dagger = (T_e)^*$. Recall that the complete norm on this algebra of operators is

$$\|T\| = \max\{\|T\|_{op}, \|T^\dagger\|_{op}\} = \max\{\|T\|_{op}, \|(T_e)^*\|_{op}\}.$$

Since $\mathcal{E}$ is a Jörgens algebra, we define the Fredholm operators with respect to $\mathcal{E}$, denoted $\Phi_{\mathcal{E}}$, as the set of operators in $\mathcal{E}$ that are invertible modulo finite rank operators in $\mathcal{E}$ (Definition 1.6.)

We restate Theorem 1.7 in terms of $\mathcal{E}$.

Theorem 3.4. *Let $T \in \mathcal{E}$. Then T has a generalized inverse in $\mathcal{E}$ if and only if*

1. *There exist projections P and Q in $\mathcal{E}$ such that*

$$\mathcal{R}(P) = \mathcal{N}(T), \quad \mathcal{R}(Q) = \mathcal{R}(T); \quad \textit{and}$$

2. *$\mathcal{R}((T_e)^*) = \mathcal{N}(T)^\perp$.*

Suppose that $\mathcal{R}(T_e) \subseteq X$. Define the bounded linear operator $K : Y \longrightarrow X$ by $Ky = T_e y$. Then we have the relations $KJ = T$ and $JK = T_e$. Again, the results of B. Barnes in [B2] apply to K and J.

Theorem 3.5. *Let $T \in \mathcal{E}$ be such that $\mathcal{R}(T_e) \subseteq X$. Then $I - T \in \Phi_{\mathcal{E}}$ if and only if $I - T \in \Phi(X)$.*

Proof. Since $\mathcal{E} = \mathcal{A}(X, Y^*)$, $I - T \in \Phi_{\mathcal{E}}$ if and only if the following three things occur: $I - T \in \Phi(X)$, $I - T^\dagger = I - (T_e)^* \in \Phi(Y^*)$ and $\iota(I - T) + \iota(I - (T_e)^*) = 0$ [B1, Theorem 2.5]. By [B2, Theorem 6], $I - T \in \Phi(X)$ if and only if $I - T_e \in \Phi(Y)$ and under these conditions $\iota(I - T) = \iota(I - T_e)$. So if $I - T \in \Phi(X)$, $I - T_e \in \Phi(Y)$; thus $I - (T_e)^* \in \Phi(Y^*)$ and $\iota(I - T_e) + \iota(I - (T_e)^*) = 0$. Therefore we have the necessary conditions for $I - T \in \Phi_{\mathcal{E}}$. □

If $I - T \in \Phi(X)$ then $I - T$ has a generalized inverse in $\mathcal{E}$. As before, more can be said in general.

Theorem 3.6. *Let $T \in \mathcal{E}$ be such that $\mathcal{R}(T_e) \subseteq X$. The operator $I - T$ has a generalized inverse in $\mathcal{E}$ if and only if $I - T$ has a generalized inverse in $B(X)$.*

Proof. Let $I - W$ be a generalized inverse of $I - T$ where $W \in B(X)$. From Lemma 3.1, $I - (W - I)T$ is also a generalized inverse of $I - T$. By [B2, Theorem 4], $I - T_e$ has a generalized inverse $I - V$ where $V = J(W - I)K$.

Let $x \in X$, $\alpha \in Y^*$ be arbitrary. Then, recalling the definition of the bilinear form and that $KJ = T$, we have

$$\begin{aligned}
\langle [I - (W - I)T]x, \alpha \rangle &= \langle (I - WT + T)x, \alpha \rangle \\
&= \langle x, \alpha \rangle - \langle (WT - T)x, \alpha \rangle \\
&= \alpha(Jx) - \alpha((JWT - JT)x) \\
&= \alpha(Jx) - \alpha(J(W - I)Tx) \\
&= \alpha(Jx) - \alpha(J(W - I)KJx) \\
&= \alpha(Jx) - \alpha(VJx) \\
&= \alpha(Jx - VJx) \\
&= \alpha((I - V)Jx) \\
&= \langle x, (I - V)^* \alpha \rangle.
\end{aligned}$$

Thus $I - V$ is an extension of $I - (W - I)T$ so $I - (W - I)T \in \mathcal{E}$ is a generalized inverse of $I - T$. □

Corollary 3.7. *Let $T \in \mathcal{E}$ be such that $\mathcal{R}(T_e) \subseteq X$. If $I - T$ has a generalized inverse, then*

1. $\mathcal{N}(I-T)^{\perp} = \mathcal{R}(I-T^{\dagger}) = \mathcal{R}(I-T_e^*)$*;*
2. $\mathcal{R}(I-T) = {}^{\perp}\mathcal{N}(I-T^{\dagger}) = {}^{\perp}\mathcal{N}(I-T_e^*)$*;*
3. $\mathcal{R}(I-T^{\dagger}) = \mathcal{R}(I-T_e^*)$ *is $\mathcal{X}$-closed.*

Under these conditions, $I-T$ has a generalized inverse if and only if $I-T_e$ does [B2, Theorem 4]. If Y is a Hilbert space, $I-T_e$ having a generalized inverse is equivalent to $\mathcal{R}(I-T_e)$ being closed. But by [B2, Theorem 5], $\mathcal{R}(I-T_e)$ is closed if and only if $\mathcal{R}(I-T)$ is closed. Therefore we have the following corollary.

Corollary 3.8. *Suppose X is a Banach space and Y is a Hilbert space with X dense in Y and continuous embedding $J : X \hookrightarrow Y$. Consider the Extension Algebra $\mathcal{E}$ as above and T in $\mathcal{E}$ such that $\mathcal{R}(T_e) \subseteq X$. Then $I-T$ has a generalized inverse in $\mathcal{E}$ if and only if $\mathcal{R}(I-T)$ is closed.*

4 Commutants and Drazin Inverses

Let X be a Banach space. Recall that for any subset A of $B(X)$, the commutant of A, denoted A', is the set of all operators in $B(X)$ that commute with every element of A. The double commutant of A, denoted A'', is the set of all operators in $B(X)$ that commute with every element of A'. Clearly $A \subseteq A''$ and both A' and A'' are Banach algebras containing the identity operator.

Throughout this section we are concerned with the commutant and double commutant of an operator $T \in B(X)$, denoted $\{T\}'$ and $\{T\}''$, respectively. One can ask when an operator T has a generalized inverse that is either in $\{T\}'$ or $\{T\}''$. It turns out that generalized inverses in these algebras are closely related to Drazin inverses. An operator $D \in B(X)$ is a Drazin inverse of $T \in B(X)$ if $TD = DT$, $D = TD^2$ and $T^k = T^{k+1}D$ for some nonnegative integer k. The smallest such k for which the equation holds is called the index of T.

Following the convention that for an operator $T \in B(X)$, $T^0 = I$, the identity operator, there are two interesting chains of subspaces:

$$\begin{gathered} \{0\} = \mathcal{N}(T^0) \subseteq \mathcal{N}(T) \subseteq \mathcal{N}(T^2) \subseteq \cdots ; \text{ and} \\ X = \mathcal{R}(T^0) \supseteq \mathcal{R}(T) \supseteq \mathcal{R}(T^2) \supseteq \cdots . \end{gathered}$$

The *ascent* of an operator T is the smallest nonnegative integer such that $\mathcal{N}(T^k) = \mathcal{N}(T^{k+1})$. If no such number exists, the ascent is infinite. The *descent* of an operator T is the smallest nonnegative integer such that $\mathcal{R}(T^k) = \mathcal{R}(T^{k+1})$. If no such number exists, we say that the descent is infinite.

Recall that the resolvent set of an operator T, denoted $\rho(T)$, is the set of all complex numbers λ for which $(\lambda - T)^{-1}$ exists. In other words, $\rho(T) = \mathbb{C} \backslash \sigma(T)$. The resolvent operator (or just resolvent) $R_\lambda : \rho(T) \longrightarrow B(X)$ is defined as the function that sends λ to $(\lambda - T)^{-1}$. The resolvent operator is holomorphic on $\rho(T)$ and is very useful in spectral theory because a holomorphic functional calculus can be obtained [Co, Section VII.4].

If $\lambda_o \in \mathbb{C}$ is an isolated point of $\sigma(T)$, one can find disjoint open sets U_1 and U_2 of the complex plane such that $\lambda_o \in U_1$ and $\sigma(T)\setminus \{\lambda_o\} \subset U_2$. Then there is a function f that is holomorphic on an open set $U = U_1 \cup U_2$ such that $f \equiv 1$ on U_1 and $f \equiv 0$ on U_2. If γ is a positively oriented curve about λ_o such that $\sigma(T)\setminus\{\lambda_o\}$ is outside of the curve, $f(T) \in B(X)$ becomes

$$f(T) := \frac{1}{2\pi i}\int_\gamma f(\lambda)R_\lambda \, d\lambda = \frac{1}{2\pi i}\int_\gamma (\lambda - T)^{-1}\, d\lambda.$$

This operator $f(T)$ is called the spectral projection (spectral idempotent, Riesz idempotent) associated with λ_o. Clearly it is a projection since $f(\lambda)f(\lambda) = f(\lambda)$ for all $\lambda \in U$.

We can now discuss generalized inverses in $\{T\}'$ and $\{T\}''$.

Theorem 4.1. *Let X be a Banach space and $T \in B(X)$ be such that T is not invertible. Then the following are equivalent:*

1. *$\mathcal{R}(T)$ is closed and $X = \mathcal{N}(T) \oplus \mathcal{R}(T)$;*
2. *T has a generalized inverse in $\{T\}'$;*
3. *T has a generalized inverse in $\{T\}''$;*
4. *T has a Drazin inverse of index 1;*
5. *0 is an isolated point of the spectrum of T and if P is the spectral projection associated with $\{0\}$, then P has the property $PT = TP = 0$;*
6. *0 is an isolated point of the spectrum of T and the spectral projection associated with $\{0\}$ is the continuous projection onto $\mathcal{N}(T)$ along $\mathcal{R}(T)$.*

Proof. (1) $\Longleftrightarrow$ (2): This follows from the proof of Theorem IV.12.9 of [TL]. The projections P and Q onto $\mathcal{N}(T)$ and $\mathcal{R}(T)$, respectively, and the generalized inverse S built from these projections satisfy $P = I - ST$ and $Q = TS$. If (1) is true, $Q = I - P$ so $ST = TS$. If (2) is true, then $Q = I - P$ and so $X = \mathcal{N}(T) \oplus \mathcal{R}(T)$.

(1) $\Longrightarrow$ (3): We have $X = \mathcal{N}(T) \oplus \mathcal{R}(T)$. Let Q be the continuous projection onto $\mathcal{N}(T)$ along $\mathcal{R}(T)$. Suppose $B \in \{T\}'$. Then for all $x \in \mathcal{N}(T)$, $BTx = TBx = 0$ so $Bx \in \mathcal{N}(T)$. Now let $x \in X$ be arbitrary. Then x has the representation $x = x_1 + Tx_2$, where $x_1 \in \mathcal{N}(T)$ and $x_2 \in X$. Then we have the following:

$$\begin{aligned} BQx &= Bx_1 \text{ and} \\ QBx &= QBx_1 + QBTx_2 \\ &= QBx_1 + QTBx_2 \\ &= Bx_1. \end{aligned}$$

Thus for all $B \in \{T\}'$, $BQ = QB$ and therefore $B(I-Q) = (I-Q)B$. Also, let S be the generalized inverse in $\{T\}'$ as constructed in Theorem IV.12.9 of [TL] (using $X = \mathcal{N}(T) \oplus \mathcal{R}(T)$). Then if T_0 is the restriction of T to $\mathcal{R}(T)$, T_0^{-1} exists and is continuous with $S = T_0^{-1}(I-Q)$. Since B commutes with $I-Q$ for all $B \in \{T\}'$, if B_0 is the restriction of B to $\mathcal{R}(T)$ we have B_0 commutes with T_0 and thus with T_0^{-1}. Thus,

$$\begin{aligned} SB &= T_0^{-1}(I-Q)B \\ &= T_0^{-1}B(I-Q) \\ &= BT_0^{-1}(I-Q) \\ &= BS \end{aligned}$$

and so $S \in \{T\}''$.

(3) $\Longrightarrow$ (2): Clear since $\{T\}'' \subseteq \{T\}'$.

(2) $\Longrightarrow$ (4): Clear by the definitions if we also note that when $S \in \{T\}'$ is a generalized inverse of T then $\widehat{S} = STS \in \{T\}'$ is also a generalized inverse of T such that $\widehat{S}T\widehat{S} = \widehat{S}$.

(4) $\Longrightarrow$ (2): Follows from the definition of Drazin inverses.

(4) $\Longrightarrow$ (6): Since T is not invertible, $0 \in \sigma(T)$. By Theorem 4 of [K] the ascent and descent of T are both 1. Since the ascent and descent of T is finite, 0 is a pole of R_λ [TL, Theorem V.10.2]. Thus $X = \mathcal{N}(T) \oplus \mathcal{R}(T)$ and $\mathcal{R}(T)$ is closed by Theorem V.6.2 of [TL]. Let Q be the projection onto $\mathcal{N}(T)$ along $\mathcal{R}(T)$. Let P be the spectral projection associated with $\{0\}$. Since $\{0\}$ is an isolated point of the spectrum, we can let γ be a positively oriented circle about 0 with radius small enough so that $\sigma(T)\backslash\{0\}$ is outside of γ. Thus we have

$$P = \frac{1}{2\pi i}\int_\gamma (\lambda - T)^{-1}\, d\lambda.$$

For all $x \in \mathcal{N}(T)$, $(\lambda - T)x = \lambda x$. For all $\lambda \in \gamma \subseteq \rho(T)$ and all $x \in \mathcal{N}(T)$, $(\lambda - T)^{-1}x = \frac{1}{\lambda}x$. We then have

$$\begin{aligned} Px &= \left(\frac{1}{2\pi i}\int_\gamma (\lambda - T)^{-1}\, d\lambda\right)x \\ &= \left(\frac{1}{2\pi i}\int_\gamma \frac{1}{\lambda}\, d\lambda\right)x \\ &= x \quad \text{ for all } x \in \mathcal{N}(T). \end{aligned}$$

Consequently, $\mathcal{R}(Q) = \mathcal{N}(T) \subseteq \mathcal{R}(P) = \mathcal{N}(I-P)$. Thus $(I-P)Q = 0$ so $Q = PQ$. Since $QT = TQ$, $QP = PQ$. Clearly $\mathcal{N}(P) \subseteq \mathcal{N}(QP) = \mathcal{N}(Q)$. Also, $\mathcal{R}(P) \cap \mathcal{R}(Q) = \mathcal{R}(PQ) = \mathcal{R}(Q)$. So $\mathcal{R}(P) \subseteq \mathcal{R}(Q)$ and thus $P = Q$.

(4) $\Longrightarrow$ (5): From the proof of (4) $\Longrightarrow$ (6), we saw that $TQ = QT = 0$ but $Q = P$ so the results holds.

(5) $\Longrightarrow$ (4): Since $\{0\}$ is an isolated point of $\sigma(T)$, 0 is a pole of R_λ of order 1 [Co, Prop. VII.6.12]. Then by Theorem V.10.1 of [TL], the ascent and descent of T are equal to 1. Thus Theorem 4 of [K] gives us that T has a Drazin inverse of index 1.

(6) $\Longrightarrow$ (1): Clear. $\square$

Note: Drazin inverses are unique. So by the above theorem, when T had a generalized inverse in $\{T\}'$ (or $\{T\}''$), it is the unique such generalized inverse.

5 Examples

Because of the work by Jörgens, many of the examples of Jörgens algebras involve Banach spaces of functions where the bilinear form is the integral of the two functions. Thus many examples involve integral and convolution operators.

Example 5.1. Consider the Jörgens algebra $\mathcal{A}(\ell^p, \ell^t)$ as discussed in [B3] with the measure μ being counting measure on $\mathbb{N}$. So we have $1 \leq p < s \leq \infty$ and $\frac{1}{s} + \frac{1}{t} = 1$ ($t = 1$ when $s = \infty$) and bilinear form $\langle \xi, \eta \rangle = \sum_{k=1}^{\infty} \xi_k \eta_k$.

Define $T \in B(\ell^p)$ by

$$T(\xi_1, \xi_2, \xi_3, \ldots) = (\xi_1, 0, \xi_2, 0, \xi_3, 0, \ldots)$$

and so

$$\langle T\xi, \eta \rangle = \sum_{k=1}^{\infty} \xi_k \eta_{2k-1}.$$

Clearly T is an isometry so $\mathcal{N}(T) = \{0\}$ and we have $\mathcal{R}(T) = \{\xi \in \ell^p | \xi_{2k} = 0 \text{ for all } k \in \mathbb{N}\}$ which is closed. The projection $Q \in B(\ell^p)$ onto $\mathcal{R}(T)$ is

$$Q(\xi_1, \xi_2, \xi_3, \ldots) = (\xi_1, 0, \xi_3, 0, \ldots).$$

The operator $T \in \mathcal{A}(\ell^p, \ell^t)$ with $T^\dagger \in B(\ell^t)$ is defined by

$$T^\dagger(\eta_1, \eta_2, \eta_3, \ldots) = (\eta_1, \eta_3, \eta_5, \ldots)$$

and clearly $\langle T\xi, \eta \rangle = \langle \xi, T^\dagger \eta \rangle$. Also, $\mathcal{R}(T^\dagger) = \ell^t$ and so is $\mathcal{X}$-closed. The projection $Q \in \mathcal{A}$ with $Q^\dagger(\eta_1, \eta_2, \eta_3, \ldots) = (\eta_1, 0, \eta_3, 0, \ldots)$. Therefore T has a generalized inverse $S \in \mathcal{A}$ with

$$S(\xi_1, \xi_2, \xi_3, \ldots) = (\xi_1, \xi_3, \xi_5, \ldots).$$

One can define operators similar to above even more generally. Let $\{e_k\}_{k \geq 1}$ denote the standard Schauder basis for ℓ^p (or ℓ^t) having one in the k-th place and zeros elsewhere. Let D and R be nonempty subsets of $\mathbb{N}$ with $card(D) =$

$card(R)$. Let $\phi : D \longrightarrow R$ be an injective map onto R. For the above example, $D = \mathbb{N}$, $R = \{n \in \mathbb{N} \mid n \text{ is odd}\}$ and $\phi(n) = 2n - 1$. For ease of notation, let $\psi : R \longrightarrow D$ be the inverse of ϕ and let α be the triple $\alpha = (D,\ R,\ \phi)$. Define T_α by

$$T_\alpha(e_k) = \begin{cases} e_{\phi(k)} & \text{if } k \in D \\ 0 & \text{if } k \notin D. \end{cases}$$

Clearly $T_\alpha \in B(\ell^p)$ with $\|T_\alpha\|_{op} = 1$. If we define the operator $T_\alpha^\dagger$ on ℓ^t by

$$T_\alpha^\dagger(e_k) = \begin{cases} e_{\psi(k)} & \text{if } k \in R \\ 0 & \text{if } k \notin R \end{cases}$$

then clearly $T_\alpha^\dagger \in B(\ell^t)$ with $\|T_\alpha^\dagger\|_{op} = 1$. Also,

$$\langle T_\alpha \xi, \eta \rangle = \sum_{k \in D} \xi_k \eta_{\phi(k)} = \sum_{k \in R} \xi_{\psi(k)} \eta_k = \langle \xi, T_\alpha^\dagger \eta \rangle$$

for all $\xi \in \ell^p$ and $\eta \in \ell^t$; consequently $T_\alpha \in \mathcal{A}(\ell^p, \ell^t)$. The projection P onto $\mathcal{N}(T_\alpha)$ and the projection Q onto $\mathcal{R}(T_\alpha)$ are defined by

$$Pe_k = \begin{cases} e_k & \text{if } k \in D \\ 0 & \text{if } k \notin D \end{cases}, \qquad Qe_k = \begin{cases} e_k & \text{if } k \in R \\ 0 & \text{if } k \notin R. \end{cases}$$

It is easy to check that P and Q are in $\mathcal{A}$ with $P^\dagger$ and $Q^\dagger$ having the same definitions on ℓ^t. Also, since

$$\mathcal{N}(T_\alpha) = \big\{ \{\xi_k\} \in \ell^p \mid \xi_k = 0 \text{ for } k \in D \big\}$$

and

$$\mathcal{R}(T_\alpha^\dagger) = \big\{ \{\xi_k\} \in \ell^t \mid \xi_k = 0 \text{ for } k \notin D \big\}$$

we have $\mathcal{N}(T_\alpha)^\perp = \mathcal{R}(T_\alpha^\dagger)$ and so Theorem 1.7 applies. Indeed, the generalized inverse of T_α has the same definition as $T_\alpha^\dagger$ but on ℓ^p.

If one did not want a partial isometry, let $\{x_n\}$ be a sequence that is bounded from above and also away from zero. Let β be the quadruple $\beta = (D, R, \phi, \{x_n\})$. Then T_β is defined as in T_α but for $k \in D$, define instead $T_\beta e_k = x_k e_{\phi(k)}$. Then $T_\beta^\dagger e_k = x_{\psi(k)} e_{\psi(k)}$ for $k \in R$. A generalized inverse S_β of T_β in $\mathcal{A}(\ell^p, \ell^t)$ would be defined by

$$S_\beta e_k = \begin{cases} \frac{1}{x_{\psi(k)}} e_{\psi(k)} & \text{if } k \in R \\ 0 & \text{if } k \notin R \end{cases} \text{ with } S_\beta^\dagger e_k = \begin{cases} \frac{1}{x_k} e_{\phi(k)} & \text{if } k \in D \\ 0 & \text{if } k \notin D. \end{cases}$$

Example 5.2. Consider the Banach space ℓ^p for $1 \le p \le 2$. This space has an inner product

$$(\xi, \eta) = \sum_{k=1}^{\infty} \xi_k \overline{\eta}_k.$$

Recall that for $p < q \leq \infty$, $\ell^p \subseteq \ell^q$ with $\|\xi\|_q \leq \|\xi\|_p$. Now suppose q is the conjugate exponent of p. Then for $1 \leq p \leq 2$, $p \leq q$. As a consequence of Hölder's Inequality,

$$\begin{aligned} |\,(\xi,\eta)\,| = \left|\sum_{k=1}^{\infty} \xi_k \overline{\eta}_k\right| &\leq \sum_{k=1}^{\infty} |\xi_k \eta_k| \\ &= \|\xi\eta\|_1 \\ &\leq \|\xi\|_p \|\eta\|_q \\ &\leq \|\xi\|_p \|\eta\|_p. \end{aligned}$$

Therefore the inner product is bounded.

Consider the operators T, T_α and T_β in $B(\ell^p)$ defined in Example 5.1. The operators are in $\mathcal{B}$, with

$$\begin{aligned} T^*\eta &= (\eta_1, \eta_3, \eta_5, \ldots), \\ T_\alpha^*\eta &= \begin{cases} e_{\psi(k)} & \text{if } k \in R \\ 0 & \text{if } k \notin R \end{cases} \\ \text{and } T_\beta^*\eta &= \begin{cases} \overline{x}_{\psi(k)} e_{\psi(k)} & \text{if } k \in R \\ 0 & \text{if } k \notin R. \end{cases} \end{aligned}$$

T has a Moore-Penrose inverse $S = T^* \in \mathcal{B}$ and T_α has a Moore-Penrose inverse $S_\alpha = T_\alpha^* \in \mathcal{B}$. The generalized inverse S_β defined in Example 5.1 of T_β is also Moore-Penrose inverse in $\mathcal{B}$ with

$$S_\beta^* e_k = \begin{cases} \frac{1}{\overline{x}_k} e_{\phi(k)} & \text{if } k \in D \\ 0 & \text{if } k \notin D. \end{cases}$$

Example 5.3. Consider $X = L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$ and the bilinear form

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x) g(x)\, dx.$$

Let $f \in L^1 \cap L^2$ and define the convolution operator T_f by

$$(T_f g)(x) = (f * g)(x) = \int_{-\infty}^{\infty} f(x-y) g(y)\, dy.$$

$T_f \in B(L^\infty \cap L^2)$ since $\|f * g\|_\infty \leq \|f\|_1 \|g\|_\infty$ and $\|f * g\|_2 \leq \|f\|_1 \|g\|_2$ for all $g \in L^\infty \cap L^2$ [HR]. The operator $T_f \in \mathcal{A}(X,X)$ with $T_f^\dagger = T_h$, where $h(x) = f(-x)$.

Also, X is dense in $Y = L^2(\mathbb{R})$ with continuous embedding the inclusion map. So $T_f \in \mathcal{E}$ with

$$(T_f)_e g(x) = \int_{-\infty}^{\infty} f(x-y) g(y)\, dy,\ g \in L^2(\mathbb{R}).$$

For any $g \in L^2(\mathbb{R})$,

$$\begin{aligned}\|(T_f)_e g\|_\infty &= \operatorname*{ess\,sup}_{x\in\mathbb{R}} \left| \int_{-\infty}^{\infty} f(x-y)g(y)\,dy \right| \\ &\leq \operatorname*{ess\,sup}_{x\in\mathbb{R}} \int_{-\infty}^{\infty} |f(x-y)g(y)|\,dy \\ &\leq \|f\|_2\|g\|_2\end{aligned}$$

by Hölder's Inequality and invariance of the measure. Thus $(T_f)_e$ maps L^2 functions to $L^\infty \cap L^2$ and so we have $\mathcal{R}((T_f)_e) \subseteq X$. Thus by Corollary 3.8, for any such operator such that $\mathcal{R}(I - T_f)$ is closed, $I - T_f$ will have a generalized inverse in $\mathcal{E}$.

Example 5.4. Let $X = C[0,1]$ and $Y = L^2[0,1]$. Consider the integral operators T_k on $C[0,1]$ with continuous kernel. $T_k \in \mathcal{E}$ with

$$(T_k)_e f(x) = \int_0^1 k(x,y)f(y)\,dy,\ \ f \in L^2[0,1].$$

Let $\{x_n\}$ be a sequence in $[0,1]$ such that $x_n \longrightarrow x_0$ and let $f \in L^2[0,1]$ be arbitrary. Then $f \in L^1[0,1]$ and since k is continuous on $[0,1]\times[0,1]$, $k(x_n,y)f(y)$ and $k(x_0,y)f(y)$ are L^1 functions for all $n \in \mathbb{N}$ and almost all $y \in [0,1]$. Also, $k(x_n,y)f(y) \longrightarrow k(x_0,y)f(y)$ pointwise and $|k(x_n,y)f(y)| \leq \|k\|_u|f(y)|$ for almost all $y \in [0,1]$. Thus, by the Lebesgue Dominated Convergence Theorem,

$$\begin{aligned}\lim_{n\longrightarrow\infty}(T_k)_e f(x_n) &= \lim_{n\longrightarrow\infty}\int_0^1 k(x_n,y)f(y)\ dy \\ &= \int_0^1 k(x_0,y)f(y)\ dy \\ &= (T_k)_e f(x_0).\end{aligned}$$

Thus $(T_k)_e f \in C[0,1]$ for all $f \in L^2[0,1]$. Operators such as T_k are compact, thus $I - T_k \in \Phi(X)$. By Theorem 3.5, $I - T_k \in \Phi_{\mathcal{E}}$ and thus $I - T_k$ has a generalized inverse in $\mathcal{E}$.

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