On the hp finite element approximation of systems of singularly perturbed reaction-diffusion equations

C. Xenophontos
Department of Mathematics and Statistics
University of Cyprus
P.O. BOX 20537
Nicosia 1678
CYPRUS
and
L. Oberbroeckling
Department of Mathematical Sciences
Loyola College in Maryland
4501 N. Charles Street
Baltimore, MD 21210
USA

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Abstract

We consider the approximation of systems of singularly perturbed reaction-diffusion equations, with the finite element method. The solution to such problems contains, in general, boundary layers which overlap and interact, and the numerical approximation must take this into account in order for the resulting scheme to converge uniformly with respect to the singular perturbation parameters. In this article, we focus on the case when the singular perturbation parameters are *equal* and adapt the analysis of the corresponding scalar problem from [9], to construct an hp finite element scheme which includes elements of size $O(\varepsilon p)$ near the boundary, where ε is the singular perturbation parameter and p is the degree of the approximating polynomials. We show that under the assumption of analytic input data, the method yields *exponential* rates of convergence, independently of ε. Numerical computations supporting the theory are also presented.

1 Introduction

The numerical solution of singularly perturbed problems has been studied extensively over the last two decades (see, e.g., the books [10], [11], [13] and the references therein). The main difficulty in these problems is the presence of *boundary layers* in the solution, whose accurate approximation, independently of the singular perturbation parameter(s), is of utmost importance in order for the overall quality of the approximate solution to be good. In the context of the Finite Element Method (FEM), the robust approximation of boundary layers requires either the use of the h version on non-uniform meshes (such as the Shishkin [19] or Bakhvalov [1] mesh), or the use of the high order p and hp versions on specially designed (variable) meshes [18]. In both cases, the a-priori knowledge of the position of the layers is taken into account, and mesh-degree combinations can be chosen for which uniform error estimates can be established [3], [9], [18].

In recent years researchers have turned their attention to *systems* of singularly perturbed problems which have two (or more) overlapping boundary layers, such as the problem considered below: Find $\overrightarrow{u}$ such that

$$L\overrightarrow{u} \equiv \begin{bmatrix} -\varepsilon_1^2 \frac{d^2}{dx^2} & & 0 \\ & \ddots & \\ 0 & & -\varepsilon_m^2 \frac{d^2}{dx^2} \end{bmatrix} \overrightarrow{u} + A\overrightarrow{u} = \overrightarrow{f} \quad \text{in} \quad \Omega = (0,1), \tag{1}$$

$$\overrightarrow{u}(0) = \overrightarrow{u}(1) = \overrightarrow{0}, \tag{2}$$

where $0 < \varepsilon_1 \leq \varepsilon_2 \leq ... \leq \varepsilon_m \leq 1$,

$$A = \begin{bmatrix} a_{11}(x) & ... & a_{1m}(x) \\ \vdots & & \vdots \\ a_{m1}(x) & ... & a_{mm}(x) \end{bmatrix}, \quad \overrightarrow{f}(x) = \begin{bmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{bmatrix}. \tag{3}$$

The data $\{\varepsilon_i\}_{i=1}^m$, A and $\overrightarrow{f}$ are given, and the unknown solution is $\overrightarrow{u}(x) = [u_1(x), ..., u_m(x)]^T$. The functions $a_{ij}(x)$ satisfy

$$a_{ij}(x) \leq 0, \quad i \neq j \quad \forall\, x \in \overline{\Omega}, \tag{4}$$

$$\min_{\overline{\Omega}} \{a_{11}(x) + ... + a_{1m}(x), a_{21}(x) + ... + a_{2m}(x), ..., \ a_{m1}(x) + a_{mm}(x)\} \geq \alpha^2 > 0, \tag{5}$$

for some $\alpha \in \mathbb{R}$. (This guarantees that A is invertible and $\|A^{-1}\|$ is bounded [2].)

The presence of the small parameters ε_i in (1) causes the solution $\overrightarrow{u}$ to contain boundary layers near the endpoints of Ω, which, in general, overlap and interact. To illustrate this, we consider the case $m = 2$ with $\varepsilon_1 = \varepsilon$, $\varepsilon_2 = \mu$, and distinguish three separate cases, depending on the relationship between ε and μ, as follows:

Case 1: $0 < \varepsilon < \mu << 1$

In this case, both components of the solution will have a boundary layer of width $O(|\mu \ln \mu|)$, while $u_1(x)$ will have an additional sublayer of width $O(|\varepsilon \ln \varepsilon|)$. This is illustrated in figure 1 below, in which the two components of the solution $\overrightarrow{u}$ are shown, for the case

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \quad \overrightarrow{f}(x) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \varepsilon = 10^{-7/2}, \quad \mu = 10^{-2}.$$

Case 2: $0 < \varepsilon < 1, \mu = 1$

In this case, only the first component of the solution will have a boundary layer of width $O(|\varepsilon \ln \varepsilon|)$, as is illustrated in figure 2 below, which corresponds to

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \quad \overrightarrow{f}(x) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \varepsilon = 10^{-2}, \quad \mu = 1.$$

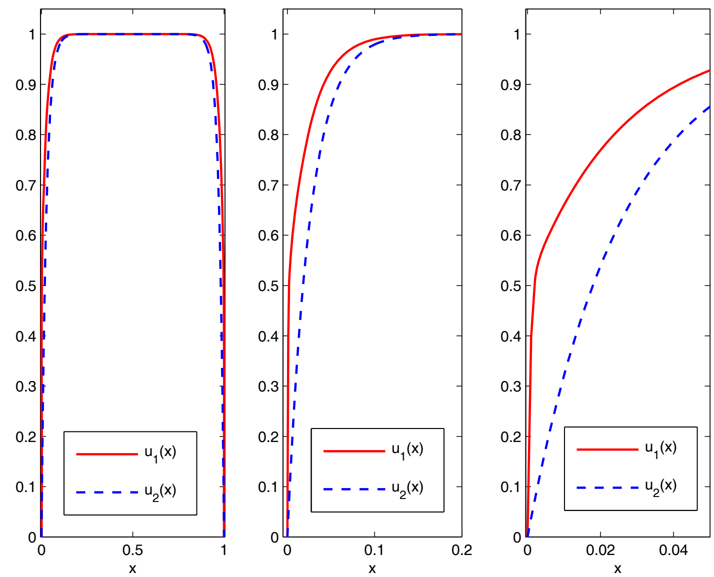

Figure 1: The exact solution for Case 1 with $\varepsilon = 10^{-7/2}, \mu = 10^{-2}$.

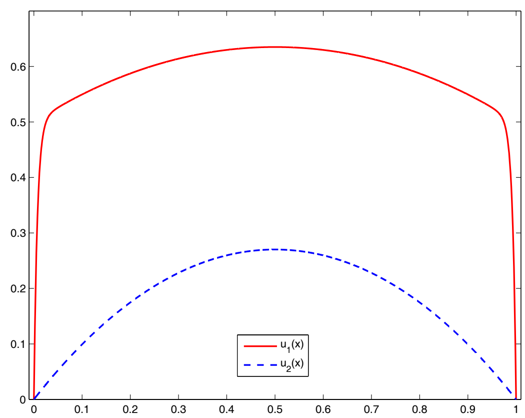

Figure 2: The exact solution for Case 2 with $\varepsilon = 10^{-2}, \mu = 1$.

The case $0 < \varepsilon << \mu \leq 1$, i.e. ε is much smaller than μ which is not that small, is qualitatively the same as Case 2 because then $u_1(x)$ will have a boundary layer of width $O(|\varepsilon \ln \varepsilon|)$, while the boundary layer of width $O(|\mu \ln \mu|)$ that $u_i(x), i = 1, 2$ will have would not be pronounced.

Case 3: $0 < \varepsilon = \mu < 1$

In this case, both components of the solution will have a boundary layer of width $O(|\varepsilon \ln \varepsilon|)$, as is illustrated in figure 3 below which corresponds to

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \quad \overrightarrow{f}(x) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \varepsilon = \mu = 10^{-2}.$$

The problem corresponding to Case 1 was studied by Matthews et al. [7, 8], Madden and Stynes [6] and by Linß and Madden [4, 5] in the context of finite differences, and by Linß and Madden

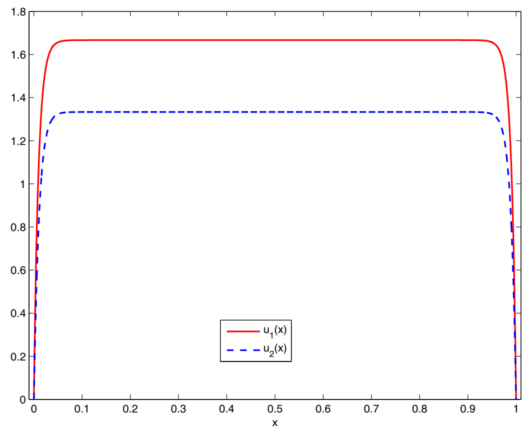

Figure 3: The exact solution for Case 3 with $\varepsilon = \mu = 10^{-2}$.

[3] in the context of the h version of the FEM with piecewise linear basis functions. In [22], the same problem was considered in a numerical study in which particular attention was paid to the high order versions of the FEM and an hp FEM on a variable mesh was proposed, which produced exponential rates of convergence like the ones obtained for the scalar analog of (1), (2) studied in [9] and [18]. The analysis of this hp scheme for Case 1, and the proof of exponential convergence appears in [23]. The problem corresponding to Case 2, was considered by Matthews et al. [7, 8] and by Tamilselvan et al. [21], in the context of finite differences, and is currently being investigated in the context of the hp version of the FEM [24]. This case also corresponds to the fourth order singularly perturbed scalar problem studied in [14]. (See also [15], [16].)

In the present article we focus on the problem corresponding to Case 3, with m equations and $\varepsilon_i = \varepsilon, i = 1, ..., m$, and extend the analysis of [9] for the analogous scalar problem, to show that under the assumption of analytic input data, the hp version of the FEM on the variable three element mesh $\Delta = \{0, \kappa p\varepsilon, 1 - \kappa p\varepsilon\}, \kappa \in \mathbb{R}^+$ converges at an *exponential* rate (in the energy norm defined in eq. (14) below) as the polynomial degree of the approximating basis functions $p \to \infty$. In addition to extending the results of [9] to systems, our proof does not use Gauss-Lobatto interpolants (like the one in [9]), but rather we achieve the desired result using the approximation theory from [17] with integrated Legendre polynomials, something that is of interest in its own right.

The rest of the paper is organized as follows: In Section 2 we present the model problem and discuss the properties of its solution. In Section 3 we present the finite element formulation and the design of the hp scheme we will be considering, along with our main result of exponential convergence. In Section 4 we present the results of some numerical computations for two model problems, and in Section 5 we summarize our conclusions.

In what follows, the space of squared integrable functions on an interval $\Omega \subset \mathbb{R}$ will be denoted by $L^2(\Omega)$, with associated inner product

$$(u, v)_\Omega := \int_\Omega uv.$$

We will also utilize the usual Sobolev space notation $H^k(\Omega)$ to denote the space of functions on Ω with $0, 1, 2, ..., k$ generalized derivatives in $L^2(\Omega)$, equipped with norm and seminorm $\|\cdot\|_{k,\Omega}$ and

$|\cdot|_{k,\Omega}$, respectively. For vector functions $\overrightarrow{u} = [u_1(x), ..., u_m(x)]^T$, we will write

$$\|\overrightarrow{u}\|_{k,\Omega}^2 = \|u_1\|_{k,\Omega}^2 + ... + \|u_m\|_{k,\Omega}^2 .$$

We will also use the space

$$H_0^1(\Omega) = \left\{u \in H^1(\Omega) : u|_{\partial\Omega} = 0\right\},$$

where $\partial\Omega$ denotes the boundary of Ω. Finally, the letter C will be used to denote a generic positive constant, independent of any discretization or singular perturbation parameters and possibly having different values in each occurrence.

2 The Model Problem and its Regularity

We consider the model problem described by (1)–(5) above, taking $\varepsilon_i = \varepsilon, \forall i = 1, ..., m$, which allows us to express it in vector form as: Find $\overrightarrow{u}$ such that

$$L\overrightarrow{u} := -\varepsilon^2 \overrightarrow{u}'' + A\overrightarrow{u} = \overrightarrow{f} \quad \text{in} \quad \Omega = (0,1) \tag{6}$$

$$\overrightarrow{u}(0) = \overrightarrow{u}(1) = \overrightarrow{0}. \tag{7}$$

In addition to (4), (5), we also assume that the functions $a_{ij}(x)$ and $f_i(x)$ are analytic on $\overline{\Omega}$ and that there exist constants C_f, γ_f, C_a, $\gamma_a > 0$ such that

$$\left\|f_i^{(n)}\right\|_{\infty,\Omega} \leq C_f \gamma_f^n n! \quad \forall\, n \in \mathbb{N}_0,\ i = 1, ..., m \tag{8}$$

$$\left\|a_{ij}^{(n)}\right\|_{\infty,\Omega} \leq C_a \gamma_a^n n! \quad \forall\, n \in \mathbb{N}_0,\ i,j = 1, ..., m. \tag{9}$$

As usual, we cast the problem (6), (7) into an equivalent weak formulation, which reads: Find $\overrightarrow{u} \in \left[H_0^1(\Omega)\right]^m$ such that

$$B(\overrightarrow{u}, \overrightarrow{v}) = F(\overrightarrow{v}) \quad \forall\, \overrightarrow{v} \in \left[H_0^1(\Omega)\right]^m, \tag{10}$$

where

$$B(\overrightarrow{u}, \overrightarrow{v}) = \varepsilon^2 \sum_{i=1}^{m} \left(u_i', v_i'\right)_\Omega + \sum_{i=1}^{m}\sum_{j=1}^{m} \left(a_{ij} u_j, v_i\right)_\Omega, \tag{11}$$

$$F(\overrightarrow{v}) = \sum_{i=1}^{m} \left(f_i, v_i\right)_\Omega . \tag{12}$$

From (5), we get that for any $x \in \overline{\Omega}$,

$$\overrightarrow{\xi}^T A \overrightarrow{\xi} \geq \alpha^2 \overrightarrow{\xi}^T \overrightarrow{\xi} \quad \forall\, \overrightarrow{\xi} \in \mathbb{R}^m, \tag{13}$$

and it follows that the bilinear form $B(\cdot,\cdot)$ is coercive with respect to the *energy norm*

$$\|\overrightarrow{u}\|_{E,\Omega}^2 := \varepsilon^2 |\overrightarrow{u}|_{1,\Omega}^2 + \alpha^2 \|\overrightarrow{u}\|_{0,\Omega}^2, \tag{14}$$

i.e.,

$$B\left(\overrightarrow{u},\overrightarrow{u}\right)\geq \|\overrightarrow{u}\|_{E,\Omega}^{2} \quad \forall\, \overrightarrow{u}\in \left[H_0^1\left(\Omega\right)\right]^m. \tag{15}$$

This, along with the continuity of $B\left(\cdot,\cdot\right)$ and $F\left(\cdot\right)$, imply the unique solvability of (10). We also have the *a priori* estimate

$$\|\overrightarrow{u}\|_{E,\Omega}\leq \frac{1}{\alpha}\left\|\overrightarrow{f}\right\|_{0,\Omega}. \tag{16}$$

We now present results on the regularity of the solution to (6), (7). Note that by the analyticity of a_{ij} and f_i, we have that u_i are analytic. Moreover, we have the following theorem.

Theorem 1. *Let $\overrightarrow{u}$ be the solution to* (6), (7) *with $0<\varepsilon\leq 1$. Then there exist constants C and $K>0$, independent of ε, such that*

$$\left\|u_i^{(n)}\right\|_{0,\Omega}\leq CK^n\max\{n,\varepsilon^{-1}\}^n \quad \forall\, n\in\mathbb{N}_0,\ i=1,...,m. \tag{17}$$

Proof. The proof follows essentially the proof of Theorem 1 from [9], and is by induction on n. The cases $n=0$ and $n=1$ follow directly from (16), so we assume (17) holds for $0\leq\nu\leq n+1$ and show that it holds for $\nu=n+2$.

Let $i\in\{1,...,m\}$ be arbitrary. From (6), we have

$$-\varepsilon_i^2 u_i''(x)+\sum_{j=1}^{m}a_{ij}(x)u_j(x)=f_i(x)$$

and hence

$$-\varepsilon^2 u_i^{(n+2)}(x)=f_i^{(n)}(x)-\sum_{j=1}^{m}\left(a_{ij}(x)u_j(x)\right)^{(n)}=f_1^{(n)}-\sum_{j=1}^{m}\sum_{\nu=0}^{n}\binom{n}{\nu}a_{ij}^{(\nu)}u_j^{(n-\nu)}.$$

Using the induction hypothesis, eqs. (8),(9) and the facts $\|f_i\|_{0,\Omega}\leq\|f_i\|_{\infty,\Omega}$, $n!\leq 2n^n$, we get

$$\begin{aligned}
\varepsilon^2\left\|u_i^{(n+2)}\right\|_{0,\Omega} &\leq \left\|f_i^{(n)}\right\|_{0,\Omega}+\sum_{j=1}^{m}\sum_{\nu=0}^{n}\binom{n}{\nu}\left\|a_{ij}^{(\nu)}\right\|_{0,\Omega}\left\|u_j^{(n-\nu)}\right\|_{0,\Omega}\\
&\leq C_f\gamma_f^n n!+m\sum_{\nu=0}^{n}\binom{n}{\nu}C_a\gamma_a^\nu \nu! CK^{n-\nu}\max\{n-\nu,\varepsilon^{-1}\}^{n-\nu}\\
&\leq C_f\gamma_f^n n!+mK^nCC_a\sum_{\nu=0}^{n}\frac{n!}{(n-\nu)!}\gamma_a^\nu K^{-\nu}\max\{n-\nu,\varepsilon^{-1}\}^{n-\nu}\\
&\leq 2C_f\gamma_f^n\max\{n,\varepsilon^{-1}\}^n+mK^nCC_a\sum_{\nu=0}^{n}\frac{n!}{(n-\nu)!}\left(\frac{\gamma_a}{K}\right)^\nu\max\{n,\varepsilon^{-1}\}^{n-\nu}.
\end{aligned}$$

Using $\frac{n!}{(n-\nu)!}\leq n^\nu$ and choosing K so that $K>\gamma_a$, we get

$$
\begin{aligned}
\varepsilon^2 \left\| u_i^{(n+2)} \right\|_{0,\Omega} &\leq 2C_f \gamma_f^n \max\{n, \varepsilon^{-1}\}^n + mK^n C C_a \sum_{\nu=0}^{n} n^\nu \left(\frac{\gamma_a}{K}\right)^\nu \max\{n,\ \varepsilon^{-1}\}^{n-\nu} \\
&\leq 2C_f \gamma_f^n \max\{n, \varepsilon^{-1}\}^n + mK^n C C_a \sum_{\nu=0}^{\infty} \left(\frac{\gamma_a}{K}\right)^\nu \max\{n,\ \varepsilon^{-1}\}^{n} \\
&\leq 2C_f \gamma_f^n \max\{n, \varepsilon^{-1}\}^n + mCC_a K^n \frac{1}{1-\frac{\gamma_a}{K}} \max\{n, \varepsilon^{-1}\}^n \\
&\leq CK^{n+2} \max\{n, \varepsilon^{-1}\}^n \left[2C_f K^{-2} \left(\frac{\gamma_f}{K}\right)^n + mC_a K^{-2} \frac{1}{1-\frac{\gamma_a}{K}} \right].
\end{aligned}
$$

If, in addition, $K > \max\{2+m, \gamma_f, C_f, C_a, \gamma_a\}$, the expression in brackets above is bounded by 1 for all $n \in \mathbb{N}$, and we obtain

$$
\left\| u_i^{(n+2)} \right\|_{0,\Omega} \leq CK^{n+2} \max\{n+2, \varepsilon^{-1}\}^{n+2}
$$

as desired. □

For ease of notation, let

$$
B = A^{-1} = \begin{bmatrix} \beta_{11} & \dots & \beta_{1m} \\ \vdots & & \vdots \\ \beta_{m1} & & \beta_{mm} \end{bmatrix}.
$$

We will now obtain a decomposition for the solution $\overrightarrow{u}$ into a smooth (asymptotic) part, two boundary layer parts and a remainder as follows:

$$
\overrightarrow{u} = \overrightarrow{w} + A^- \overrightarrow{u}^- + A^+ \overrightarrow{u}^+ + \overrightarrow{r}. \tag{18}
$$

This decomposition is obtained by inserting the formal ansatz

$$
\overrightarrow{u}(x) \sim \sum_{i=0}^{\infty} \varepsilon^i \overrightarrow{u}_i(x), \tag{19}
$$

into the differential equation (6), and equating like powers of ε, so that we can define the smooth part $\overrightarrow{w}$ as

$$
\overrightarrow{w}(x) := \sum_{i=0}^{M} \varepsilon^{2i} \overrightarrow{u}_{2i}, \tag{20}
$$

where the terms $\overrightarrow{u}_{2i}$ are defined recursively by

$$
\overrightarrow{u}_0 = B \overrightarrow{f}, \tag{21}
$$

$$
\overrightarrow{u}_{2i} = B \left(\overrightarrow{u}_{2i}\right)'' ,\ i = 0, 2, 4, \dots \tag{22}
$$

A calculation shows that

$$
L(\overrightarrow{u} - \overrightarrow{w}) = \varepsilon^{2M+2} \left(\overrightarrow{u}_{2M}\right)'' , \tag{23}
$$

hence, as $\varepsilon \to 0$, $\overrightarrow{w}(x)$ defined by (20) satisfies the differential equation, but not the boundary conditions. To correct this we introduce *boundary layer functions* $\overrightarrow{u}^+$ and $\overrightarrow{u}^-$ by

$$\begin{aligned} L\overrightarrow{u}^- &= \overrightarrow{0} \text{ in } \Omega & \qquad L\overrightarrow{u}^+ &= \overrightarrow{0} \text{ in } \Omega \\ \overrightarrow{u}^-(0) &= [1,...,1]^T \in \mathbb{R}^m & \qquad \overrightarrow{u}^+(0) &= \overrightarrow{0} \\ \overrightarrow{u}^-(1) &= \overrightarrow{0} & \qquad \overrightarrow{u}^+(1) &= [1,...,1]^T \in \mathbb{R}^m. \end{aligned} \tag{24}$$

In order to satisfy the boundary conditions we set

$$A^- = \begin{bmatrix} -w_1(0) & & 0 \\ & \ddots & \\ 0 & & -w_m(0) \end{bmatrix},$$

and

$$A^+ = \begin{bmatrix} -w_1(1) & & 0 \\ & \ddots & \\ 0 & & -w_m(1) \end{bmatrix}.$$

Finally, we define $\overrightarrow{r}$ by

$$\begin{aligned} L\overrightarrow{r} &= \varepsilon^{2M+2} \left(\overrightarrow{u}_{2M}\right)'' \\ \overrightarrow{r}(0) &= \overrightarrow{r}(1) = \overrightarrow{0}. \end{aligned} \tag{25}$$

The following results analyze the behavior of the terms in the decomposition of $\overrightarrow{u}$.

Lemma 2. *Let $\overrightarrow{u}_{2i}$ be defined as in (21)–(22). Then there exist constants $C, K_1, K_2 > 0$ depending only on A and $\overrightarrow{f}$ such that for any $i, n \in \mathbb{N}_0$*

$$\left\| \left(\overrightarrow{u}_{2i}\right)^{(n)} \right\|_{\infty,\Omega} \leq C K_1^{2i} K_2^n (2i)! n!$$

Proof. Since the functions $\beta_{ij}, i, j = 1, ..., m$ of $B = A^{-1}$ are analytic on $\overline{\Omega}$, there exists a complex neighborhood $G \subset \mathbb{C}$ of $\overline{\Omega} = [0,1]$ on which all β_{ij} are holomorphic and bounded. Also, since the functions $f_i, i = 1, ..., m$ of $\overrightarrow{f}$ are analytic on $\overline{\Omega}$, we can assume, without loss of generality, that they are also holomorphic on G. Thus, by following the proof of Lemma 8 from [9], we can show that there exist constants C, K_1, $K_2 > 0$ depending only on G and $\|B\|_{\infty,G}$ such that for all $i, n \in \mathbb{N}_0$

$$\left\| \left(\overrightarrow{u}_{2i}\right)^{(n)} \right\|_{\infty,\Omega} \leq C K_1^{2i} K_2^n (2i)! n! \left\| \overrightarrow{u}_0 \right\|_{\infty,G} .$$

Since $\|\overrightarrow{u}_0\|_{\infty,G} \leq C$, we have the desired result. ☐

Theorem 3. *There exist constants C, $\overline{K}_1$, $\overline{K}_2 \in \mathbb{R}^+$ depending only on $\overrightarrow{f}$ and A such that if $0 < 2M\varepsilon\overline{K}_1 \leq 1$, then $\overrightarrow{w}(x)$ given by (20), satisfies*

$$\left\| \overrightarrow{w}^{(n)} \right\|_{\infty,\Omega} \leq C \overline{K}_2^n n! \;\forall\, n \in \mathbb{N}_0. \tag{26}$$

Proof. This is a direct consequence of Lemma 2 above; see also Theorem 6 from [9] for the proof of this result for the scalar problem. ☐

Remark 1. *By the previous result, we see that A^+ and A^- in the decomposition* (18) *for $\overrightarrow{u}$ are bounded independently of ε.*

We now derive bounds on the boundary layer part $\overrightarrow{u}^-$. The bounds for $\overrightarrow{u}^+$ can be derived in an analogous way.

Theorem 4. *Let $\overrightarrow{u}^-$ be the solution of* (24)*. Then there exist constants C, $K > 0$ independent of ε and n such that for any $x \in \overline{\Omega}$, $n \in \mathbb{N}_0$,*

$$\left| \left(u_i^-\right)^{(n)}(x) \right| \leq CK^n e^{-x\alpha/\varepsilon} \max\{n, \varepsilon^{-1}\}^n, \quad i = 1, ..., m. \tag{27}$$

Proof. The proof is by induction on n. The cases $n = 0$ and $n = 1$ were shown in [6] for the case of 2 equations, and the extension to m equations is straight forward. Thus, assume (27) holds for $0 \leq k \leq n+1$ and establish it for $n+2$. Let $i \in \{1, ..., m\}$ be arbitrary. We have from (24)

$$\varepsilon^2 \left(u_i^-\right)''(x) = \sum_{j=1}^{m} a_{ij}(x) u_j^-(x).$$

Using (9) and the induction hypothesis, we get

$$\begin{aligned}
\varepsilon^2 \left| \left(u_i^-\right)^{(n+2)}(x) \right| &\leq \sum_{j=1}^{m}\sum_{k=0}^{n} \binom{n}{k} \left| a_{ij}^{(k)}(x) \right| \left| \left(u_j^-\right)^{(n-k)}(x) \right| \\
&\leq \sum_{k=0}^{n} \binom{n}{k} m C_a \gamma_a^k k! C K^{n-k} e^{-x\alpha/\varepsilon} \max\{n-k, \varepsilon^{-1}\}^{n-k} \\
&\leq mCK^n C_a \sum_{k=0}^{n} \frac{n!}{(n-k)!} \left(\frac{\gamma_a}{K}\right)^k e^{-x\alpha/\varepsilon} \max\{n-k, \varepsilon^{-1}\}^{n-k} \\
&\leq mCK^{n+2} \left[\frac{C_a K^{-2}}{1 - \frac{\gamma_a}{K}}\right] e^{-x\alpha/\varepsilon} \max\{n, \varepsilon^{-1}\}^n.
\end{aligned}$$

We may choose $K > \max\{1, C_a, \gamma_a\}$ so that the expression in brackets above is bounded by 1 hence

$$\left| \left(u_1^-\right)^{(n+2)}(x) \right| \leq CK^{n+2} e^{-x\alpha/\varepsilon} \max\{n+2, \varepsilon^{-1}\}^{n+2},$$

as desired. □

Finally, we investigate the remainder $\overrightarrow{r}$.

Theorem 5. *There are constants C, K_1, $K_2 > 0$ depending only on the input data such that the remainder $\overrightarrow{r}$ defined by* (25) *satisfies*

$$\left\| \overrightarrow{r}^{(n)} \right\|_{0,\Omega} \leq CK_2^2 \varepsilon^{2-n} \left(2M\varepsilon K_1\right)^{2M}, \ n = 0, 1. \tag{28}$$

Proof. Since $\overrightarrow{r}$ satisfies (25), the a priori estimate (16) gives

$$\| \overrightarrow{r} \|_{E,\Omega} \leq \frac{1}{\alpha} \left\| \varepsilon^{2M+2} \left(\overrightarrow{u}_{2M}\right)'' \right\|_{0,\Omega}.$$

By Lemma 2,

$$\left\|\varepsilon^{2M+2}\left(\overrightarrow{u}_{2M}\right)''\right\|_{\infty,\Omega} \leq CK_1^{2M}K_2^2(2M)!,$$

thus

$$\|\overrightarrow{r}\|_{E,\Omega} \leq CK_2^2\varepsilon^{2M+2}K_1^{2M}(2M)! \leq CK_2^2\varepsilon^{2M+2}K_1^{2M}(2M)^{2M},$$

from which the result follows. $\square$

3 The Finite Element Method

For the discretization of (10), we choose a finite dimensional subspace S_N of $H_0^1(\Omega)$ and solve the problem: Find $\overrightarrow{u}_N \in [S_N]^m$ such that

$$B\left(\overrightarrow{u}_N, \overrightarrow{v}\right) = F\left(\overrightarrow{v}\right) \quad \forall\, \overrightarrow{v} \in [S_N]^m. \tag{29}$$

The unique solvability of the discrete problem (29) follows from (13) and (15); by the well-known orthogonality relation, we have

$$\|\overrightarrow{u} - \overrightarrow{u}_N\|_E \leq \inf_{\overrightarrow{v} \in [S_N]^m} \|\overrightarrow{u} - \overrightarrow{v}\|_E. \tag{30}$$

The subspace S_N is chosen as follows: Let $\Delta = \{0 = x_0 < x_1 < ... < x_{\mathcal{M}} = 1\}$ be an arbitrary partition of $\Omega = (0,1)$ and set

$$I_j = (x_{j-1}, x_j),\ h_j = x_j - x_{j-1},\ j = 1, ..., \mathcal{M}.$$

Also, define the master (or standard) element $I_{ST} = (-1,1)$, and note that it can be mapped onto the j^{th} element I_j by the linear mapping

$$x = Q_j(t) = \frac{1}{2}(1-t)\,x_{j-1} + \frac{1}{2}(1+t)\,x_j.$$

With $\Pi_p(I_{ST})$ the space of polynomials of degree $\leq p$ on I_{ST}, we define our finite dimensional subspaces as

$$S_N \equiv S^{\overrightarrow{p}}(\Delta) = \left\{u \in H_0^1(\Omega) : u\left(Q_j(t)\right) \in \Pi_{p_j}(I_{ST}),\, j = 1, ..., \mathcal{M}\right\}$$

and

$$\overrightarrow{S}_0^p(\Delta) := \left[S^{\overrightarrow{p}}(\Delta) \cap H_0^1(\Omega)\right]^m, \tag{31}$$

where $\overrightarrow{p} = (p_1, ..., p_{\mathcal{M}})$ is the vector of polynomial degrees assigned to the elements.

The following approximation result from [17] will be the main tool for the analysis of the method. As mentioned earlier, the analysis for the scalar problem in [9] relied on Gauss-Lobatto interpolants (and their approximation properties), which is different from what we present in this work.

Theorem 6. *For any $u \in C^\infty\left(\overline{I}_{ST}\right)$ there exists $\mathcal{I}_p u \in \Pi_p(I_{ST})$ such that*

$$u(\pm 1) = \mathcal{I}_p u(\pm 1), \tag{32}$$

$$\|u - \mathcal{I}_p u\|_{0,I_{ST}}^2 \leq \frac{1}{p^2}\frac{(p-s)!}{(p+s)!}\left\|u^{(s+1)}\right\|_{0,I_{ST}}^2,\ \forall\, s = 0, 1, ..., p, \tag{33}$$

$$\left\|(u - \mathcal{I}_p u)'\right\|_{0,I_{ST}}^2 \leq \frac{(p-s)!}{(p+s)!}\left\|u^{(s+1)}\right\|_{0,I_{ST}}^2,\ \forall\, s = 0, 1, ..., p. \tag{34}$$

The definition below describes the mesh used for the method: If we are in the asymptotic range of p, i.e. $p \geq 1/\varepsilon$, then a single element suffices since p will be sufficiently large to give us exponential convergence without any refinement. If we are in the pre-asymptotic range, i.e. $p < 1/\varepsilon$, then the mesh consists of three elements as described below. We should point out that this is the *minimal* mesh-degree combination for attaining exponential convergence; obviously, refining within each element will retain the convergence rate but would require more degrees of freedom – one such example is the so-called *geometrically graded* mesh discussed in [9] for the scalar problem.

Definition 7. *For $\kappa > 0$, $p \in \mathbb{N}$ and $0 < \varepsilon \leq 1$, define the spaces $\overrightarrow{S}(\kappa, p)$ of piecewise polynomials by*

$$\overrightarrow{S}(\kappa, p) := \begin{cases} \overrightarrow{S}_0^p(\Delta);\ \Delta = \{0, 1\} & \text{if } \kappa p \varepsilon \geq \frac{1}{2} \\ \overrightarrow{S}_0^p(\Delta);\ \Delta = \{0, \kappa p \varepsilon, 1 - \kappa p \varepsilon, 1\} & \text{if } \kappa p \varepsilon < \frac{1}{2}. \end{cases}$$

In both cases, the polynomial degree is uniformly p on all elements.

Before we state the main theorem of the paper, we cite a useful computation.

Lemma 8. *Let $p \in \mathbb{N}, \lambda \in (0, 1]$. Then*

$$\frac{(p - \lambda p)!}{(p + \lambda p)!} \leq \left[\frac{(1 - \lambda)^{(1-\lambda)}}{(1 + \lambda)^{(1+\lambda)}} \right]^p p^{-2\lambda p} e^{2\lambda p + 1}.$$

Proof. Using Stirling's approximation

$$\sqrt{2\pi n} \left(\frac{n}{e} \right)^n e^{\frac{1}{12n+1}} \leq n! \leq \sqrt{2\pi n} \left(\frac{n}{e} \right)^n e^{\frac{1}{12n}} \leq \sqrt{2\pi n} \left(\frac{n}{e} \right)^n e$$

for the factorial (cf. [12]), we have

$$\begin{aligned} \frac{(p - \lambda p)!}{(p + \lambda p)!} &\leq \frac{\sqrt{2\pi(1-\lambda)p} \left(\frac{(1-\lambda)p}{e} \right)^{(1-\lambda)p}}{\sqrt{2\pi(1+\lambda)p} \left(\frac{(1+\lambda)p}{e} \right)^{(1+\lambda)p}} \frac{e}{e^{\frac{1}{12(1+\lambda)p+1}}} \leq \frac{[(1-\lambda)\,p]^{(1-\lambda)p}}{[(1+\lambda)\,p]^{(1+\lambda)p}} e^{2\lambda p} e^{1 - \frac{1}{12(1+\lambda)p+1}} \\ &\leq \left[\frac{(1 - \lambda)^{(1-\lambda)}}{(1 + \lambda)^{(1+\lambda)}} \right]^p p^{-2\lambda p} e^{2\lambda p} e. \end{aligned}$$

□

We now present our main result.

Theorem 9. *Let $\overrightarrow{f}$ and A be composed of functions that are analytic on $\overline{\Omega}$ and satisfy the conditions in (8),(9). Let $\overrightarrow{u} = [u_1, ..., u_m]^T$ be the solution to (6), (7). Then there exist constants $\kappa, C, \beta > 0$ depending only on $\overrightarrow{f}$ and A such that there exists $\mathcal{I}_p \overrightarrow{u} = [\mathcal{I}_p u_1, ..., \mathcal{I}_p u_m]^T \in \overrightarrow{S}(\kappa, p)$ with $\mathcal{I}_p \overrightarrow{u} = \overrightarrow{u}$ on $\partial\Omega$ and*

$$\|\overrightarrow{u} - \mathcal{I}_p \overrightarrow{u}\|_{E,\Omega}^2 \leq C p^3 e^{-\beta p}.$$

Proof.

Case 1. $\kappa p\varepsilon \geq \frac{1}{2}$ (asymptotic case), $\Delta = \{0, 1\}$

From Theorem 1 we have

$$\left\| \overrightarrow{u}^{(n)} \right\|^2_{0,\Omega} \leq CK^{2n} \max\{n, \varepsilon^{-1}\}^{2n},$$

and by Theorem 6 there exists $\mathcal{I}_p \overrightarrow{u} \in \overrightarrow{S}(\kappa, p)$ such that $\overrightarrow{u} = \mathcal{I}_p \overrightarrow{u}$ on $\partial\Omega$ and for any $s = 0, 1, ..., p$

$$\|(\overrightarrow{u} - \mathcal{I}_p \overrightarrow{u})'\|^2_{0,\Omega} \leq \frac{(p-s)!}{(p+s)!} \left\| \overrightarrow{u}^{(s+1)} \right\|^2_{0,\Omega} \leq \frac{(p-s)!}{(p+s)!} CK^{2(s+1)} \max\{s+1, \varepsilon^{-1}\}^{2(s+1)}.$$

Let $s = \lambda p$ for some $\lambda \in (0, 1]$. Then, since $p \geq 1/(2\kappa\varepsilon)$, we have

$$\max\{s+1, \varepsilon^{-1}\}^{2(s+1)} = \max\{\lambda p + 1, \varepsilon^{-1}\}^{2(\lambda p+1)} = (\lambda p + 1)^{2(\lambda p+1)}$$

which, along with Lemma 8, gives

$$\begin{aligned}
\|(\overrightarrow{u} - \mathcal{I}_p \overrightarrow{u})'\|^2_{0,\Omega} &\leq \frac{(p-\lambda p)!}{(p+\lambda p)!} CK^{2(\lambda p+1)} (\lambda p + 1)^{2(\lambda p+1)} \\
&\leq \left[\frac{(1-\lambda)^{(1-\lambda)}}{(1+\lambda)^{(1+\lambda)}} \right]^p p^{-2\lambda p} e^{2\lambda p+1} CK^{2(\lambda p+1)} (\lambda p + 1)^{2(\lambda p+1)} \\
&\leq CeK^2 \left[\frac{(1-\lambda)^{(1-\lambda)}}{(1+\lambda)^{(1+\lambda)}} (eK)^{2\lambda} \right]^p (\lambda p + 1)^2 \left(\frac{1+\lambda p}{p} \right)^{2\lambda p} \\
&\leq CeK^2 p^2 \left[\frac{(1-\lambda)^{(1-\lambda)}}{(1+\lambda)^{(1+\lambda)}} (eK)^{2\lambda} \right]^p \left(\frac{1}{p} + \lambda \right)^{2\lambda p}.
\end{aligned}$$

Since $\left(\frac{1}{p} + \lambda\right)^{2\lambda p} = \lambda^{2\lambda p} \left[\left(1 + \frac{1}{\lambda p} \right)^{\lambda p} \right]^2 \leq e^2 \lambda^{2\lambda p}$, we further get

$$\|(\overrightarrow{u} - \mathcal{I}_p \overrightarrow{u})'\|^2_{0,\Omega} \leq Cp^2 \left[\frac{(1-\lambda)^{(1-\lambda)}}{(1+\lambda)^{(1+\lambda)}} (eK\lambda)^{2\lambda} \right]^p,$$

so if we choose $\lambda = (eK)^{-1} \in (0, 1)$ we have

$$\|(\overrightarrow{u} - \mathcal{I}_p \overrightarrow{u})'\|^2_{0,\Omega} \leq Cp^2 e^{-\beta_1 p}, \tag{35}$$

where $\beta_1 = |\ln q_1|$, $q_1 = \frac{(1-\lambda)^{(1-\lambda)}}{(1+\lambda)^{(1+\lambda)}} < 1$. Repeating the previous argument for the L^2 norm of $(\overrightarrow{u} - \mathcal{I}_p \overrightarrow{u})$, we get, using (34),

$$\|\overrightarrow{u} - \mathcal{I}_p \overrightarrow{u}\|^2_{0,\Omega} \leq Ce^{-\beta_1 p}. \tag{36}$$

Combining (35)–(36), and using the definition of the energy norm, we get the desired result.

Case 2. $\kappa p\varepsilon < \frac{1}{2}$ (pre-asymptotic case), $\Delta = \{0, \kappa p\varepsilon, 1 - \kappa p\varepsilon, 1\}$

The mesh consists of three elements I_i, $i = 1, 2, 3$ and we decompose $\overrightarrow{u}$ as in (18):

$$\overrightarrow{u} = \overrightarrow{w} + A^- \overrightarrow{u}^- + A^+ \overrightarrow{u}^+ + \overrightarrow{r}.$$

The expansion order M is chosen so $2M = \eta\kappa p$ where $\eta > 0$ is a fixed parameter satisfying

$$\frac{1}{2}\eta\overline{K}_1 \leq 1, \qquad \frac{1}{2}\eta K_1 =: \delta < \frac{1}{2}$$

with $\overline{K}_1$ and K_1 the constants from Theorems 3 and 5, respectively. The choice of η guarantees that as $\kappa p\varepsilon < \frac{1}{2}$, we have

$$2M\varepsilon\overline{K}_1 = \eta\kappa p\varepsilon\overline{K}_1 < \frac{1}{2}\eta\overline{K}_1 \leq 1$$

and

$$2M\varepsilon K_1 = \eta\kappa p\varepsilon K_1 < \frac{1}{2}\eta K_1 =: \delta < \frac{1}{2}.$$

Thus the assumptions of Theorem 3 are satisfied and the remainder $\overrightarrow{r}$ is small by Theorem 5 – in particular, we have

$$\left\|(\overrightarrow{r})^{(n)}\right\|_{0,\Omega} \leq C\varepsilon^{2-n}K_2^2\left(2M\varepsilon K_1\right)^{2M} \leq C\varepsilon^{2-n}\delta^{\eta\kappa p} \leq C\varepsilon^{2-n}e^{-\beta_2 p},\ n = 0,1 \tag{37}$$

where $\beta_2 = \left|\ln q_2\right|,\ q_2 = \delta^{\eta\kappa} < 1$.

We next analyze the approximation of each of the remaining three terms in the decomposition (18).

For the approximation of $\overrightarrow{w}$, we have, by Theorem 6, that there exists $\mathcal{I}_p\overrightarrow{w} \in \overrightarrow{S}(\kappa,p)$ such that $\overrightarrow{w} = \mathcal{I}_p\overrightarrow{w}$ on $\partial\Omega$ and for any $s = 0,1,...,p$

$$\left\|(\overrightarrow{w} - \mathcal{I}_p\overrightarrow{w})'\right\|_{0,\Omega}^2 \leq \frac{(p-s)!}{(p+s)!}\left\|\overrightarrow{w}^{(s+1)}\right\|_{0,\Omega}^2 \leq \frac{(p-s)!}{(p+s)!}CK^{2(s+1)}\left((s+1)!\right)^2,$$

where we used Theorem 3. Letting $s = \overline{\lambda}p$, for some $\overline{\lambda} \in (0,1]$, and using Lemma 8, we get

$$\begin{aligned}
\left\|(\overrightarrow{w} - \mathcal{I}_p\overrightarrow{w})'\right\|_{0,\Omega}^2 &\leq \left[\frac{\left(1-\overline{\lambda}\right)^{(1-\overline{\lambda})}}{\left(1+\overline{\lambda}\right)^{(1+\overline{\lambda})}}\right]^p p^{-2\overline{\lambda}p}e^{2\overline{\lambda}p+1}CK_2^{2\overline{\lambda}p+2}\left[\left(\overline{\lambda}p+1\right)^{\overline{\lambda}p+1+1/2}e^{-\overline{\lambda}p-1}\right]^2 \\
&\leq C\left(\overline{\lambda}p+1\right)^3\left[\frac{\left(1-\overline{\lambda}\right)^{(1-\overline{\lambda})}}{\left(1+\overline{\lambda}\right)^{(1+\overline{\lambda})}}\right]^p K_2^{2\overline{\lambda}p}\left(\frac{1+\overline{\lambda}p}{p}\right)^{2\lambda p} \\
&\leq C\left(\overline{\lambda}p+1\right)^3\left[\frac{\left(1-\overline{\lambda}\right)^{(1-\overline{\lambda})}}{\left(1+\overline{\lambda}\right)^{(1+\overline{\lambda})}}\right]^p K_2^{2\overline{\lambda}p}\overline{\lambda}^{2\overline{\lambda}p}\left[\left(1+\frac{1}{\overline{\lambda}p}\right)^{\overline{\lambda}p}\right]^2 \\
&\leq Cp^3\left[\frac{\left(1-\overline{\lambda}\right)^{(1-\overline{\lambda})}}{\left(1+\overline{\lambda}\right)^{(1+\overline{\lambda})}}\left(K_2\overline{\lambda}\right)^{2\overline{\lambda}}\right]^p.
\end{aligned}$$

Thus, we choose $\overline{\lambda} = 1/K_2 \in (0,1)$ and we have

$$\left\|(\overrightarrow{w} - \mathcal{I}_p\overrightarrow{w})'\right\|_{0,\Omega}^2 \leq Cp^3e^{-\beta_2 p}, \tag{38}$$

where $\beta_2 = \left|\ln q_2\right|$, $q_2 = \frac{\left(1-\overline{\lambda}\right)^{(1-\overline{\lambda})}}{\left(1+\overline{\lambda}\right)^{(1+\overline{\lambda})}} < 1$. Repeating the previous argument for the L^2 norm of $(\overrightarrow{w} - \mathcal{I}_p\overrightarrow{w})$, we get, using (34),

$$\left\|\overrightarrow{w} - \mathcal{I}_p\overrightarrow{w}\right\|_{0,\Omega}^2 \leq Cpe^{-\beta_2 p}. \tag{39}$$

We now approximate the boundary layers. We will only consider $A^-\overrightarrow{u}^-$, since $A^+\overrightarrow{u}^+$ is completely analogous, and we will concentrate on the approximation of $\overrightarrow{u}^-$, since $A^- = \text{diag}(-w_1(0), ..., -w_m(0))$ is bounded independently of ε (see Remark 1). In view of Theorem 4, we will construct separate approximations for $\overrightarrow{u}^-$ on the intervals $\widetilde{I}_1 := I_1 = [0, \kappa p\varepsilon]$, and $\widetilde{I}_2 := [\kappa p\varepsilon, 1]$. Let $i \in \{1, ..., m\}$ be arbitrary. Then, by Theorem 6 there exists $\mathcal{I}_p u_i^- \in S(\kappa, p)$ such that $\mathcal{I}_p u_i^- = u_i^-$ on $\partial\widetilde{I}_1$ and for any $s = 0, 1, ..., p$

$$\left\| \left(u_i^- - \mathcal{I}_p u_i^-\right)' \right\|_{0,\widetilde{I}_1}^2 \leq (\kappa p\varepsilon)^{2s} \frac{(p-s)!}{(p+s)!} \left\| \left(u_i^-\right)^{(s+1)} \right\|_{0,\widetilde{I}_1}^2 . \tag{40}$$

Now, by Lemma 4, we have

$$\begin{aligned} \left\| \left(u_i^-\right)^{(s+1)} \right\|_{0,\widetilde{I}_1}^2 &= \int_0^{\kappa p\varepsilon} \left| \left(u_i^-\right)^{(s+1)}(x) \right|^2 dx \\ &\leq C\kappa p\varepsilon K^{2(s+1)} \max\{s+1, \varepsilon^{-1}\}^{2(s+1)} \max_{x\in[0,\kappa p\varepsilon]} \{e^{-x\alpha/\varepsilon}\}. \end{aligned} \tag{41}$$

Since $\kappa p\varepsilon < 1/2$, i.e. $s \leq p < \frac{1}{2\kappa\varepsilon}$, we have that $\max\{s+1, \varepsilon^{-1}\}^{2(s+1)} = \varepsilon^{-2(s+1)}$ and (40), (41) give

$$\begin{aligned} \left\| \left(u_i^- - \mathcal{I}_p u_i^-\right)' \right\|_{0,\widetilde{I}_1}^2 &\leq (\kappa p\varepsilon)^{2s} \frac{(p-s)!}{(p+s)!} C\kappa p\varepsilon K^{2(s+1)} \varepsilon^{-2(s+1)} \\ &\leq CK^{2(s+1)} \kappa^{2s+1} p^{2s+1} \varepsilon^{-1} \frac{(p-s)!}{(p+s)!}. \end{aligned}$$

Choosing $s = \widetilde{\lambda} p$ for some $\widetilde{\lambda} \in (0, 1)$, and using Lemma 8, we further obtain

$$\begin{aligned} \left\| \left(u_i^- - \mathcal{I}_p u_i^-\right)' \right\|_{0,\widetilde{I}_1}^2 &\leq CK^{2(\widetilde{\lambda}p+1)} \kappa^{2\widetilde{\lambda}p+1} p^{2\widetilde{\lambda}p+1} \varepsilon^{-1} \frac{(p-\widetilde{\lambda}p)!}{(p+\widetilde{\lambda}p)!} \\ &\leq CK^{2(\widetilde{\lambda}p+1)} \kappa^{2\widetilde{\lambda}p+1} p^{2\widetilde{\lambda}p+1} \varepsilon^{-1} \left[\frac{\left(1-\widetilde{\lambda}\right)^{(1-\widetilde{\lambda})}}{\left(1+\widetilde{\lambda}\right)^{(1+\widetilde{\lambda})}} \right]^p p^{-2\widetilde{\lambda}p} e^{2\widetilde{\lambda}p+1} \\ &\leq CeK^2 \kappa p\varepsilon^{-1} \left[\frac{\left(1-\widetilde{\lambda}\right)^{(1-\widetilde{\lambda})}}{\left(1+\widetilde{\lambda}\right)^{(1+\widetilde{\lambda})}} \right]^p (Ke\kappa)^{2\widetilde{\lambda}p} \\ &\leq Cp\varepsilon^{-1} e^{-\beta_3 p}, \end{aligned} \tag{42}$$

where $\beta_3 = |\ln q_3|$, $q_3 = \frac{(1-\widetilde{\lambda})^{(1-\widetilde{\lambda})}}{(1+\widetilde{\lambda})^{(1+\widetilde{\lambda})}} < 1$, *provided* we choose $\kappa = \frac{1}{eK}$. Now, on the interval $\widetilde{I}_2 = [\kappa p\varepsilon, 1]$, u_i^- is already exponentially small, and by Lemma 4

$$\left\| \left(u_i^-\right)' \right\|_{0,\widetilde{I}_2}^2 = \int_{\kappa p\varepsilon}^1 \left| \left(u_i^-\right)' \right|^2 dx \leq C\varepsilon^{-2} (1-\kappa p\varepsilon) \max_{x\in\widetilde{I}_2} \left\{ e^{-2x\alpha/\varepsilon} \right\} \leq C\varepsilon^{-2} e^{-2\kappa p\alpha}.$$

Thus, we approximate u_i^- by its linear interpolant $\mathcal{I}_1 u_i^-$, and we have

$$\left\| \left(u_i^- - \mathcal{I}_1 u_i^-\right)' \right\|_{0,\widetilde{I}_2}^2 \leq \left\| \left(u_i^-\right)' \right\|_{0,\widetilde{I}_2}^2 + \left\| \left(\mathcal{I}_1 u_i^-\right)' \right\|_{0,\widetilde{I}_2}^2 \leq C\varepsilon^{-2} e^{-2\kappa p\alpha},$$

which along with (42) give

$$\left\|\left(u_i^- - \mathcal{I}_p u_i^-\right)'\right\|_{0,\Omega}^2 \leq Cp\varepsilon^{-2}e^{-\beta_4 p}, \tag{43}$$

for some $\beta_4 > 0$ independent of ε. Repeating the previous arguments for the L^2 norm of $\left(u_i^- - \mathcal{I}_p u_i^-\right)$, we get

$$\left\|u_i^- - \mathcal{I}_p u_i^-\right\|_{0,\Omega}^2 \leq Ce^{-\beta_5 p}, \tag{44}$$

for some $\beta_5 > 0$, independent of ε. Using the same techniques, similar bounds can be obtained for $\overrightarrow{u}^+$.

Combining (37), (39), (43), (44) and the analogous bounds for $\overrightarrow{u}^+$, we have

$$\begin{aligned}
\|\overrightarrow{u} - \mathcal{I}_p \overrightarrow{u}\|_{0,\Omega}^2 &= \left\|\left(\overrightarrow{w} + A^-\overrightarrow{u}^- + A^+\overrightarrow{u}^+ + \overrightarrow{r}\right) - \left(\mathcal{I}_p\overrightarrow{w} + A^-\mathcal{I}_p\overrightarrow{u}^- + A^+\mathcal{I}_p\overrightarrow{u}^+ + \overrightarrow{r}\right)\right\|_{0,\Omega}^2 \\
&\leq \|\overrightarrow{w} - \mathcal{I}_p\overrightarrow{w}\|_{0,\Omega}^2 + C\left\{\left\|\overrightarrow{u}^- - \mathcal{I}_p\overrightarrow{u}^-\right\|_{0,\Omega}^2 + \left\|\overrightarrow{u}^+ - \mathcal{I}_p\overrightarrow{u}^+\right\|_{0,\Omega}^2\right\} + \|\overrightarrow{r}\|_{0,\Omega}^2 \\
&\leq Cpe^{-\beta p},
\end{aligned}$$

for some $\beta > 0$, independent of ε. Similarly,

$$\begin{aligned}
|u_i - \mathcal{I}_p u_i|_{1,\Omega}^2 &\leq |w_i - \mathcal{I}_p w_i|_{1,\Omega}^2 + C\left\{\left|u_i^- - \mathcal{I}_p u_i^-\right|_{1,\Omega}^2 + \left|u_i^+ - \mathcal{I}_p u_i^+\right|_{1,\Omega}^2\right\} + |r_i|_{1,\Omega}^2 \\
&\leq C\varepsilon^{-2}p^3e^{-\beta p},
\end{aligned}$$

so that

$$\|\overrightarrow{u} - \mathcal{I}_p\overrightarrow{u}\|_{E,\Omega}^2 = \varepsilon^2\sum_{i=1}^{m}|u_i - \mathcal{I}_p u_i|_{1,\Omega}^2 + \alpha^2\|\overrightarrow{u} - \mathcal{I}_p\overrightarrow{u}\|_{0,\Omega}^2 \leq Cp^3e^{-\beta p}$$

as desired. □

Remark 2. *In contrast to the analysis for the scalar problem carried out in* [9]*, our approach allows for the choice of κ in the definition of the mesh, to be made more specific, even when the data of the problem is not constant. As was shown in the proof of the above theorem, κ can be chosen based on the constant of analyticity of the input data.*

Using Theorem 9 and the quasioptimality result (30) we have the following.

Corollary 10. *Let $\overrightarrow{u}$ be the solution to* (6),(7) *and let $\overrightarrow{u}_{FE} \in \overrightarrow{S}_0^p(\Delta)$ be the solution to* (29)*. Then exist constants $\kappa, C, \sigma > 0$ depending* only *on the input data $\overrightarrow{f}$ and A such that*

$$\|\overrightarrow{u} - \overrightarrow{u}_{FE}\|_E \leq Cp^{3/2}e^{-\sigma p}.$$

4 Numerical Experiments

In this section we present the results of numerical computations for systems of 2 equations (i.e. $m = 2$) with $\varepsilon_1 = \varepsilon_2 = \varepsilon$, having as our goal the illustration our theoretical findings; we refer the interested reader to [22] for a detailed numerical study in which several other cases are considered.

4.1 The constant coefficient case

First we consider the constant coefficient case, in which

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \overrightarrow{f}(x) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \overrightarrow{u}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

An exact solution is available, hence the computations we report are reliable. We will be plotting the percentage relative error in the energy norm, given by

$$100 \times \frac{\|\overrightarrow{u}_{EXACT} - \overrightarrow{u}_{FEM}\|_{E,\Omega}}{\|\overrightarrow{u}_{EXACT}\|_{E,\Omega}}, \tag{45}$$

versus the number of degrees of freedom N, on log-log and semilog scales.

Figure 4 shows the performance of the *hp* version on the 3 element mesh for different values of ε and we observe that the method not only does not deteriorate as $\varepsilon \to 0$, but it actually performs better, when the error is measured in the energy norm. This suggests that there is a positive power of ε in the error estimate of Corollary 10. In fact, for the corresponding scalar problem with constant coefficients and polynomial right hand side, this was shown to be true in [18], where the estimate

$$\|\overrightarrow{u}_{EXACT} - \overrightarrow{u}_{FEM}\|_{E,\Omega} \leq C\varepsilon^{1/2}\alpha^p, \;\; C \in \mathbb{R}^+, \; \alpha \in (0,1)$$

was proven. This allows for the derivation of an analogous estimate in the maximum norm which, although will not contain any positive powers of ε, will show that the method converges at an exponential rate independently of ε. (See [22] for an illustration of this for the same model problem with various values of the singular perturbation parameters.)

4.2 The variable coefficient case

Next, we consider the variable coefficient case, in which

$$A = \begin{bmatrix} 2\,(x+1)^2 & -\left(1+x^2\right) \\ -2\cos(\pi x/4) & 2.2e^{1-x} \end{bmatrix}, \overrightarrow{f}(x) = \begin{bmatrix} 2e^x \\ 10x+1 \end{bmatrix}, \overrightarrow{u}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

An exact solution is not available, and for our computations we use a reference solution obtained with a large number of degrees of freedom on a very fine mesh which includes exponential refinement near the endpoints of the domain (see [22] for more details). We are again interested in the (now estimated) percentage relative error in the energy norm, as given by (45).

In figure 5 we show the *hp* version on the 3 element mesh, for different values of ε, and we observe that the method performs better as $\varepsilon \to 0$, in the variable coefficient case as well. This, does not, strictly speaking, agree with the theory and it could very well be due to the fact that we used a reference solution instead of an exact solution for the computations. Nevertheless, the exponential convergence is visible.

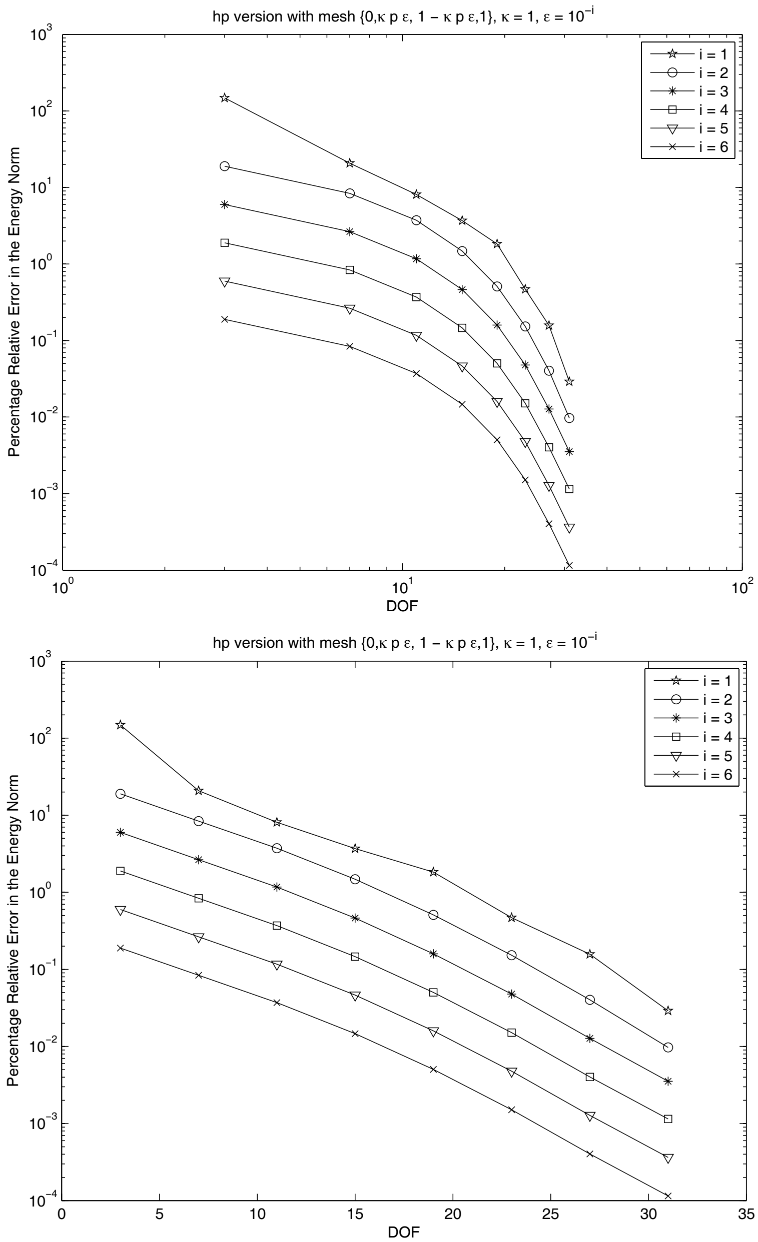

Figure 4: Energy norm convergence for the *hp* version. Top: log-log scale; Bottom: semilog scale.

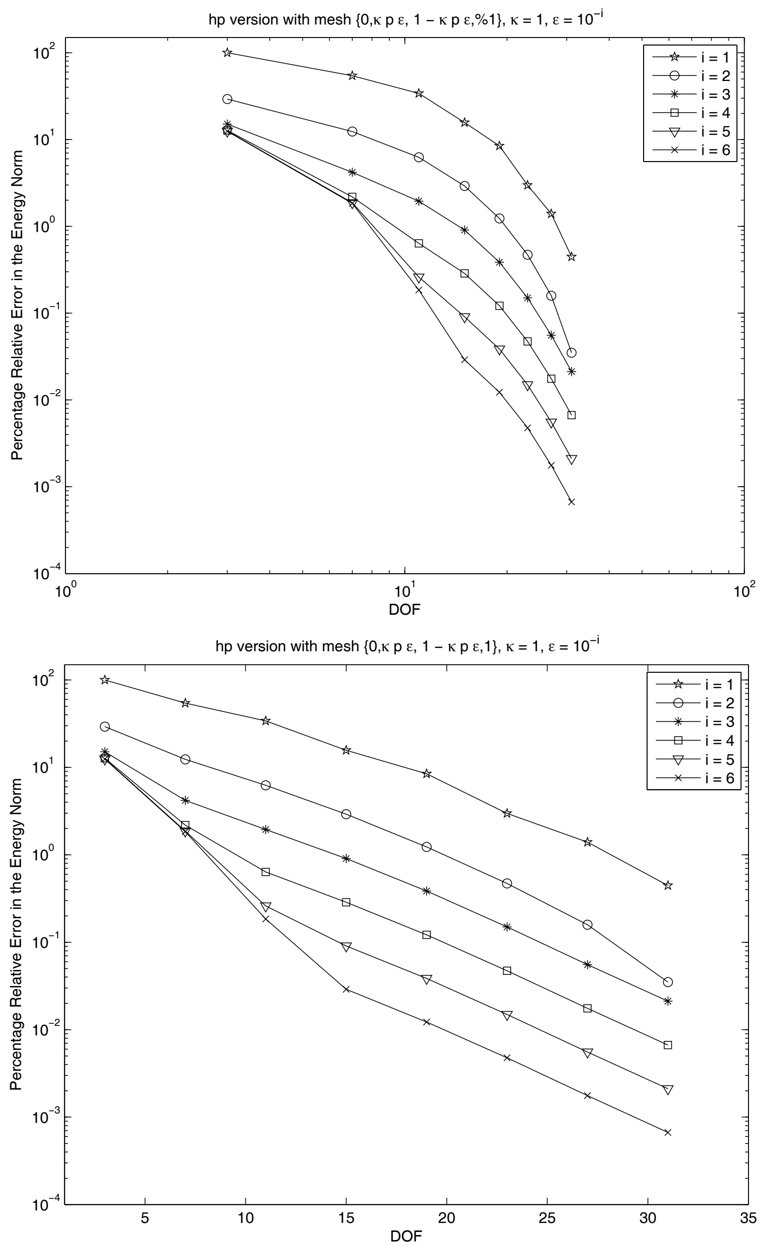

Figure 5: Energy norm convergence for the *hp* version. Top: loglog scale; Bottom: semilog scale.

5 Conclusions

We have studied the finite element approximation of systems of singularly perturbed reaction-diffusion equations in which the singular perturbation parameter is the same. We showed that under the assumption of analytic input data, the *hp* version on the variable three element mesh $\{0, \kappa p\varepsilon, 1-\kappa p\varepsilon, 1\}$ yields exponential convergence as $p \to \infty$, independently of ε, when the error is measured in the energy norm. The constant κ in the mesh was shown to depend on the constant of analyticity of the input data.

Through two numerical experiments, we verified the established exponential rate and observed that, for the problems under consideration, the method performs better as $\varepsilon \to 0$, as was the case for the corresponding scalar problem studied in [18].

The case when the singular perturbation parameters is treated in [23], [24].

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