Robust exponential convergence of *hp*-FEM for singularly perturbed reaction diffusion systems with multiple scales

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Abstract

We consider the approximation of a coupled system of two singularly perturbed reaction-diffusion equations, with the finite element method. The solution to such problems contains boundary layers which overlap and interact, and the numerical approximation must take this into account in order for the resulting scheme to converge uniformly with respect to the singular perturbation parameters. We propose and analyze an *hp* finite element scheme which includes elements of size $O(\varepsilon p)$ and $O(\mu p)$ near the boundary, where ε, μ are the singular perturbation parameters and p is the degree of the approximating polynomials. We show that under the assumption of analytic input data, the method yields *exponential* rates of convergence, independently of ε and μ.

1 Introduction

The numerical solution of singularly perturbed problems has been studied extensively over the last decade (see, e.g., the books [10], [11], [14] and the references therein). The main difficulty in these problems is the presence of *boundary layers* in the solution, whose accurate approximation, independently of the singular perturbation parameter(s), is of utmost importance in order for the overall quality of the approximate solution to be good. In the context of the Finite Element Method

(FEM), the robust approximation of boundary layers requires either the use of the h version on non-uniform meshes (such as the Shishkin [17] or Bakhvalov [1] mesh), or the use of the high order p and hp versions on specially designed (variable) meshes [16]. In both cases, the a-priori knowledge of the position of the layers is taken into account, and mesh-degree combinations can be chosen for which uniform error estimates can be established [3], [9], [16].

In recent years researchers have turned their attention to *systems* of singularly perturbed problems, which have two (or more) overlapping boundary layers, such as the one considered below: Find $\vec{u}$ such that

$$L\vec{u} := \begin{bmatrix} -\varepsilon^2 \frac{d^2}{dx^2} & 0 \\ 0 & -\mu^2 \frac{d^2}{dx^2} \end{bmatrix} \vec{u} + A\vec{u} = \vec{f} \quad \text{in} \quad \Omega = (0,1) \tag{1}$$

where $0 < \varepsilon \leq \mu \leq 1$,

$$A = \begin{bmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{bmatrix}, \quad \vec{f}(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} \tag{2}$$

along with the boundary conditions on $\partial\Omega$

$$\vec{u}(0) = \vec{\gamma_0}, \quad \vec{u}(1) = \vec{\gamma_1}. \tag{3}$$

The data ε, μ, A, $\vec{f}$, $\vec{\gamma_0}$ and $\vec{\gamma_1}$ are given and the unknown solution is $\vec{u}(x) = [u_1(x), u_2(x)]^T$. Without loss of generality we will take $\vec{\gamma_0} = \vec{\gamma_1} = \vec{0}$, and in addition we will assume that

$$a_{12}(x) \leq 0, \quad a_{21}(x) \leq 0 \quad \forall\, x \in \overline{\Omega}, \tag{4}$$

$$\min_{\overline{\Omega}} \{a_{11}(x) + a_{12}(x),\ a_{21}(x) + a_{22}(x)\} \geq \alpha^2 > 0 \tag{5}$$

for some $\alpha \in \mathbb{R}$. (This guarantees that A is invertible and $\|A^{-1}\|$ is bounded [2].)

The presence of ε and μ in (1) causes the solution $\vec{u}$ to have boundary layers near the endpoints of Ω, which, in general, overlap and interact. Problems of this type arise in the modelling of turbulance in water waves [19], as well as in the finite element approximation of shells, where the singular perturbation parameters are related to the thickness t of the shell; for example, in Naghdi-type thin shell models in mechanics there is an $O(t)$ layer due to shear deformation and there is a second layer (or length scale) $O(t^\beta)$, with $\beta \in \{1/2, 1/3, 1/4\}$ (depending on the principal curvatures of the shell's midsurface), due to bending and membrane coupling [12]. The 2-scale reaction-diffusion systems we investigate in this article could be considered model problems for this situation, with $\varepsilon = t$ and $\mu = t^\beta$.

Matthews et al. [7, 8], studied the above problem for the cases $0 < \varepsilon = \mu << 1$ and $0 < \varepsilon << \mu = 1$, obtaining an approximation using finite differences which converged indepedently of ε and μ. The more general case of $0 < \varepsilon \leq \mu \leq 1$ was studied by Madden and Stynes [6] and by Linß and Madden [4, 5] in the context of finite differences, and by Linß and Madden [3] in the context of the h version of the FEM with piecewise linear basis functions. In all the works mentioned, estimates were obtained showing that the approximation converged (at the expected rate) indepedently of ε and μ.

In [20] the same problem was considered in a numerical study, where particular attention was paid to the high order p and hp versions of the FEM. In particular, an hp scheme on a 5 element variable mesh was proposed, which included elements of size $O(p\varepsilon)$ and $O(p\mu)$ near the boundary, where p

is the degree of the approximating polynomials. This scheme is the analog of the *minimal* mesh degree combination for the corresponding scalar problem considered in [9] and [16]. It was observed that this *hp* scheme produced *exponential* convergence rates, as $p \to \infty$, independently of ε and μ. Our goal in this article is to establish the observed exponential rate, in the case $0 < \varepsilon < \mu << 1$. This is, arguably, the most challenging (and interesting) case, because while both components of $\overrightarrow{u}$ will have a boundary layer of width $O(|\mu \ln \mu|)$, the first component $u_1(x)$ will have an additional sublayer of width $O(|\varepsilon \ln \varepsilon|)$. We should mention that the case $0 < \varepsilon = \mu \leq 1$ is conceptually the same as the scalar problem and its analysis appears in [21], while the case $0 < \varepsilon << \mu = 1$ is currently being investigated.

The rest of the paper is organized as follows: In Section 2 we present the model problem and discuss the properties of its solution. In Section 3 we present the finite element formulation and the design of the *hp* scheme we will be considering, along with our main result of exponential convergence. Finally, in Section 4 we summarize our conclusions.

In what follows, the space of squared integrable functions on an interval $\Omega \subset \mathbb{R}$ will be denoted by $L^2(\Omega)$, with associated inner product

$$(u, v)_\Omega := \int_\Omega uv.$$

We will also utilize the usual Sobolev space notation $H^k(\Omega)$ to denote the space of functions on Ω with $0, 1, 2, ..., k$ generalized derivatives in $L^2(\Omega)$, equipped with norm and seminorm $\|\cdot\|_{k,\Omega}$ and $|\cdot|_{k,\Omega}$, respectively. For vector functions $\overrightarrow{u} = [u_1(x), u_2(x)]^T$, we will write

$$\|\overrightarrow{u}\|_{k,\Omega}^2 = \|u_1\|_{k,\Omega}^2 + \|u_2\|_{k,\Omega}^2.$$

We will also use the space

$$H_0^1(\Omega) = \left\{ u \in H^1(\Omega) : u|_{\partial\Omega} = 0 \right\},$$

where $\partial\Omega$ denotes the boundary of Ω. Finally, the letter C will be used to denote a generic positive constant, independent of any discretization or singular perturbation parameters and possibly having different values in each occurrence.

2 The Model Problem and its Regularity

We consider the model problem (1), (3), described in the previous section, for $0 < \varepsilon < \mu << 1$, and we assume that the functions $a_{ij}(x)$ and $f_i(x)$ are analytic on $\overline{\Omega}$ and that there exist constants C_f, γ_f, C_a, $\gamma_a > 0$ such that

$$\left\| f_i^{(n)} \right\|_{\infty,\Omega} \leq C_f \gamma_f^n n! \quad \forall\, n \in \mathbb{N}_0,\ i = 1, 2, \tag{6}$$

$$\left\| a_{ij}^{(n)} \right\|_{\infty,\Omega} \leq C_a \gamma_a^n n! \quad \forall\, n \in \mathbb{N}_0,\ i, j = 1, 2. \tag{7}$$

As usual, we cast the problem (1), (3) into an equivalent weak formulation, which reads: Find $\overrightarrow{u} \in \left[H_0^1(\Omega)\right]^2$ such that

$$B(\overrightarrow{u}, \overrightarrow{v}) = F(\overrightarrow{v}) \quad \forall\, \overrightarrow{v} \in \left[H_0^1(\Omega)\right]^2, \tag{8}$$

where

$$\begin{aligned} B\left(\overrightarrow{u}, \overrightarrow{v}\right) &= \varepsilon^2 \left(u_1', v_1'\right)_\Omega + \mu^2 \left(u_2', v_2'\right)_\Omega + \left(a_{11}u_1 + a_{12}u_2, v_1\right)_\Omega + \left(a_{21}u_1 + a_{22}u_2, v_2\right)_\Omega, \quad (9) \\ F\left(\overrightarrow{v}\right) &= \left(f_1, v_1\right)_\Omega + \left(f_2, v_2\right)_\Omega . \quad (10) \end{aligned}$$

From (5), we get that for any $x \in \overline{\Omega}$,

$$\overrightarrow{\xi}^T A \overrightarrow{\xi} \geq \alpha^2 \overrightarrow{\xi}^T \overrightarrow{\xi} \quad \forall\, \overrightarrow{\xi} \in \mathbb{R}^2, \tag{11}$$

and it follows that the bilinear form $B(\cdot,\cdot)$ is coercive with respect to the *energy norm*

$$\|\overrightarrow{u}\|_{E,\Omega}^2 := \varepsilon^2 \left|u_1\right|_{1,\Omega}^2 + \mu^2 \left|u_2\right|_{1,\Omega}^2 + \alpha^2 \left(\|u_1\|_{0,\Omega}^2 + \|u_2\|_{0,\Omega}^2\right), \tag{12}$$

i.e.,

$$B\left(\overrightarrow{u}, \overrightarrow{u}\right) \geq \|\overrightarrow{u}\|_{E,\Omega}^2 \;\; \forall\, \overrightarrow{u} \in \left[H_0^1(\Omega)\right]^2 . \tag{13}$$

This, along with the continuity of $B(\cdot,\cdot)$ and $F(\cdot)$, imply the unique solvability of (8). We also have the following *a priori* estimate

$$\|\overrightarrow{u}\|_{E,\Omega} \leq \frac{1}{\alpha} \left\|\overrightarrow{f}\right\|_{0,\Omega} . \tag{14}$$

For the discretization, we choose a finite dimensional subspace S_N of $H_0^1(\Omega)$ and solve the problem: Find $\overrightarrow{u}_N \in [S_N]^2$ such that

$$B\left(\overrightarrow{u}_N, \overrightarrow{v}\right) = F\left(\overrightarrow{v}\right) \;\; \forall\, \overrightarrow{v} \in [S_N]^2 . \tag{15}$$

The unique solvability of the discrete problem (15) follows from (11) and (13), and by the well-known orthogonality relation, we have

$$\|\overrightarrow{u} - \overrightarrow{u}_N\|_{E,\Omega} \leq \inf_{\overrightarrow{v} \in [S_N]^2} \|\overrightarrow{u} - \overrightarrow{v}\|_E . \tag{16}$$

We now present results on the regularity of the solution to (1), (3). We follow the results found in [9] for the analogous scalar problem and extend them to the case of systems; we also extend some of the regularity results from [6]. Note that by the analyticity of a_{ij} and f_i, we have that u_i are analytic. Moreover, we have the following theorem.

Theorem 1. *Let $\overrightarrow{u}$ be the solution to* (1), (3) *with* $0 < \varepsilon < \mu << 1$. *Then there exist constants C and $K > 0$, independent of ε and μ, such that*

$$\left\|u_i^{(n)}\right\|_{0,\Omega} \leq C K^n \max\{n, \varepsilon^{-1}\}^n \quad \forall\, n \in \mathbb{N}_0,\; i = 1, 2. \tag{17}$$

Proof. The proof is by induction on n. The cases $n = 0$ and $n = 1$ follow directly from (14), so we assume (17) holds for $0 \leq \nu \leq n + 1$ and show that it holds for $\nu = n + 2$.

From (1), we have

$$-\varepsilon^2 u_1''(x) + a_{11}(x) u_1(x) + a_{12} u_2(x) = f_1(x)$$

and hence

$$-\varepsilon^2 u_1^{(n+2)}(x) = f_1^{(n)}(x) - (a_{11}(x)u_1(x))^{(n)} - (a_{12}(x)u_2(x))^{(n)}$$
$$= f_1^{(n)} - \sum_{\nu=0}^{n} \binom{n}{\nu} \left[a_{11}^{(\nu)} u_1^{(n-\nu)} + a_{12}^{(\nu)} u_2^{(n-\nu)}\right]$$

Using the induction hypothesis, eqs. (6),(7) and the facts $\|f_1\|_{0,\Omega} \leq \|f_1\|_{\infty,\Omega}$, $n! \leq 2n^n$, we get

$$\begin{aligned}
\varepsilon^2 \left\| u_1^{(n+2)} \right\|_{0,\Omega} &\leq \left\| f_1^{(n)} \right\|_{0,\Omega} + \sum_{\nu=0}^{n} \binom{n}{\nu} \left[\left\| a_{11}^{(\nu)} \right\|_{0,\Omega} \left\| u_1^{(n-\nu)} \right\|_{0,\Omega} + \left\| a_{12}^{(\nu)} \right\|_{0,\Omega} \left\| u_2^{(n-\nu)} \right\|_{0,\Omega} \right] \\
&\leq C_f \gamma_f^n n! + 2 \sum_{\nu=0}^{n} \binom{n}{\nu} C_a \gamma_a^\nu \nu! C K^{n-\nu} \max\{n-\nu, \varepsilon^{-1}\}^{n-\nu} \\
&\leq C_f \gamma_f^n n! + 2K^n C C_a \sum_{\nu=0}^{n} \frac{n!}{(n-\nu)!} \gamma_a^\nu K^{-\nu} \max\{n-\nu, \varepsilon^{-1}\}^{n-\nu} \\
&\leq 2C_f \gamma_f^n \max\{n, \varepsilon^{-1}\}^n + 2K^n C C_a \sum_{\nu=0}^{n} \frac{n!}{(n-\nu)!} \left(\frac{\gamma_a}{K}\right)^\nu \max\{n, \varepsilon^{-1}\}^{n-\nu}.
\end{aligned}$$

Using $\frac{n!}{(n-\nu)!} \leq n^\nu$ and choosing K so that $K > \gamma_a$, we get

$$\begin{aligned}
\varepsilon^2 \left\| u_1^{(n+2)} \right\|_{0,\Omega} &\leq 2C_f \gamma_f^n \max\{n, \varepsilon^{-1}\}^n + 2K^n C C_a \sum_{\nu=0}^{n} n^\nu \left(\frac{\gamma_a}{K}\right)^\nu \max\{n,\ \varepsilon^{-1}\}^{n-\nu} \\
&\leq 2C_f \gamma_f^n \max\{n, \varepsilon^{-1}\}^n + 2K^n C C_a \sum_{\nu=0}^{\infty} \left(\frac{\gamma_a}{K}\right)^\nu \max\{n,\ \varepsilon^{-1}\}^n \\
&\leq 2C_f \gamma_f^n \max\{n, \varepsilon^{-1}\}^n + 2C C_a K^n \frac{1}{1-\frac{\gamma_a}{K}} \max\{n, \varepsilon^{-1}\}^n \\
&\leq C K^{n+2} \max\{n, \varepsilon^{-1}\}^n \left[2C_f K^{-2} \left(\frac{\gamma_f}{K}\right)^n + 2C_a K^{-2} \frac{1}{1-\frac{\gamma_a}{K}} \right].
\end{aligned}$$

If, in addition, $K > \max\{4, \gamma_f, C_f, C_a, \gamma_a\}$, the expression in brackets above is bounded by 1 for all $n \in \mathbb{N}$, and we obtain

$$\left\| u_1^{(n+2)} \right\|_{0,\Omega} \leq C K^{n+2} \max\{n+2, \varepsilon^{-1}\}^{n+2}$$

as desired. The second component of $\overrightarrow{u}$ is shown to satisfy the desired bound in an analogous way and by noting that $\mu^{-2} < \varepsilon^{-2}$. □

For ease of notation, let

$$B = A^{-1} = \begin{bmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{bmatrix}.$$

We will now obtain a decomposition for the solution $\overrightarrow{u}$ into a smooth (asymptotic) part, two boundary layer parts and a remainder as follows:

$$\overrightarrow{u} = \overrightarrow{w} + A^- \overrightarrow{u}^- + A^+ \overrightarrow{u}^+ + \overrightarrow{r}. \tag{18}$$

This decomposition is obtained by inserting the formal ansatz

$$u_k(x) \sim \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \varepsilon^i \mu^j u_k^{(i,j)}(x),\ k = 1, 2 \tag{19}$$

into the differential equation (1), and equating like powers of ε and μ, so that we can define the smooth part $\overrightarrow{w}$ as

$$\overrightarrow{w}(x) := \sum_{i=0}^{M}\sum_{j=0}^{M} \varepsilon^{2i}\mu^{2j}\overrightarrow{u}^{(2i,2j)} \tag{20}$$

where the terms $\overrightarrow{u}^{(2i,2j)}$ are defined recursively by

$$\overrightarrow{u}^{(0,0)} = B\overrightarrow{f}, \tag{21}$$

$$\overrightarrow{u}^{(2i,0)} = B\begin{bmatrix}\left(u_1^{(2i-2,0)}\right)'' \\ 0\end{bmatrix}, \qquad \overrightarrow{u}^{(0,2j)} = B\begin{bmatrix}0 \\ \left(u_2^{(0,2j-2)}\right)''\end{bmatrix}, \tag{22}$$

$$\overrightarrow{u}^{(2i,2j)} = B\begin{bmatrix}\left(u_1^{(2i-2,2j)}\right)'' \\ \left(u_2^{(2i,2j-2)}\right)''\end{bmatrix},\ i, j = 1, 2, ... \tag{23}$$

A calculation shows that

$$L(\overrightarrow{u} - \overrightarrow{w}) = \varepsilon^{2M+2}\sum_{j=0}^{M}\mu^{2j}\begin{bmatrix}\left(u_1^{(2M,2j)}\right)'' \\ 0\end{bmatrix} + \mu^{2M+2}\sum_{i=0}^{M}\varepsilon^{2i}\begin{bmatrix}0 \\ \left(u_2^{(2i,2M)}\right)''\end{bmatrix}, \tag{24}$$

hence, as ε, $\mu \to 0$, $\overrightarrow{w}(x)$ defined by (20) satisfies the differential equation, but not the boundary conditions. To correct this we introduce *boundary layer functions* $\overrightarrow{u}^{+}$ and $\overrightarrow{u}^{-}$ by

$$\begin{aligned} L\overrightarrow{u}^{-} &= \overrightarrow{0} \text{ in } \Omega & L\overrightarrow{u}^{+} &= \overrightarrow{0} \text{ in } \Omega \\ \overrightarrow{u}^{-}(0) &= [1,1]^T & \overrightarrow{u}^{+}(0) &= \overrightarrow{0} \\ \overrightarrow{u}^{-}(1) &= \overrightarrow{0} & \overrightarrow{u}^{+}(1) &= [1,1]^T. \end{aligned} \tag{25}$$

In order to satisfy the boundary conditions we set

$$A^{-} = \begin{bmatrix}-w_1(0) & 0 \\ 0 & -w_2(0)\end{bmatrix},$$

and

$$A^{+} = \begin{bmatrix}-w_1(1) & 0 \\ 0 & -w_2(1)\end{bmatrix}$$

in (18). Finally, we define $\overrightarrow{r}$ by

$$\begin{aligned} L\overrightarrow{r} &= \varepsilon^{2M+2}\sum_{j=0}^{M}\mu^{2j}\begin{bmatrix}\left(u_1^{(2M,2j)}\right)'' \\ 0\end{bmatrix} + \mu^{2M+2}\sum_{i=0}^{M}\varepsilon^{2i}\begin{bmatrix}0 \\ \left(u_2^{(2i,2M)}\right)''\end{bmatrix} \\ \overrightarrow{r}(0) &= \overrightarrow{r}(1) = \overrightarrow{0}. \end{aligned} \tag{26}$$

The following results analyze the behavior of the terms in the decomposition of $\overrightarrow{u}$.

Lemma 2. *Let $\overrightarrow{u}^{(2i,2j)}$ be defined as in (21)–(23). Let $G \subset \mathbb{C}$ be a complex neighborhood of $\overline{\Omega} = [0,1]$. Then there exist constants K_1, $K_2 > 0$ depending only on G and $\|B\|_{\infty,G}$ such that for all $n \in \mathbb{N}_0$*

$$\left\| \left(u_l^{(2i,2j)}\right)^{(n)} \right\|_{\infty,\Omega} \leq K_1^{2i+2j} K_2^n (2i+2j)! n! \left\| \overrightarrow{u}^{(0,0)} \right\|_{\infty,G} \tag{27}$$

where i, $j \in \mathbb{N}_0$, and $l = 1, 2$.

Proof. The proof follows that of Lemma 2 from [9]. For $\delta \in (0,1]$, let $G_\delta := \{z \in G \,:\, \text{dist}(z, \partial G) \geq \delta\}$.

Claim. $\left\|u_l^{(2i,2j)}\right\|_{\infty,G_\delta} \leq \delta^{-(2i+2j)} K^{2i+2j} (2i+2j)! \left\|\overrightarrow{u}^{(0,0)}\right\|_{\infty,G}$, for $l = 1, 2$, i, $j \in \mathbb{N}_0$.

Assuming the claim holds, let $r = \min\limits_{z \in \overline{\Omega}} \{\text{dist}(z, \partial G)\}$ and $\delta = \min\{1, \frac{r}{2}\}$. Then for any $z \in \Omega$, the disk with radius δ and center z is in $G_\delta \subset G$. For any x on the circle with radius δ and center z, we have $\left|u_l^{(2i,2j)}(x)\right| \leq \left\|u_l^{(2i,2j)}\right\|_{\infty,G_\delta}$. So by Cauchy's Integral Theorem and the claim, we have for any $z \in \Omega$,

$$\begin{aligned} \left| \left(u_l^{(2i,2j)}\right)^{(n)} (z) \right| &\leq \frac{n!}{\delta^n} \left\|u_l^{(2i,2j)}\right\|_{\infty,G_\delta} \leq \frac{n!}{\delta^n} \delta^{-(2i+2j)} K^{2i+2j} (2i+2j)! \left\|\overrightarrow{u}^{(0,0)}\right\|_{\infty,G} \\ &\leq \left(\frac{K}{\delta}\right)^{2i+2j} \left(\frac{1}{\delta}\right)^n (2i+2j)! n! \left\|\overrightarrow{u}^{(0,0)}\right\|_{\infty,G}. \end{aligned}$$

We note that K is chosen such that $K^2 > 4e \|B\|_{\infty,G}$ and that δ depends on G. Thus if we let $K_1 = \dfrac{K}{\delta}$ and $K_2 = \dfrac{1}{\delta}$, the constants K_1 and K_2 depend only on G and $\|B\|_{\infty,G}$ and the lemma is proven.

It remains to prove the claim. By (21)–(23),

$$u_1^{(2i,0)}(z) = \beta_1(z) \left(u_1^{(2i-2,0)}\right)''(z), \quad u_2^{(2i,0)}(z) = \beta_3(z) \left(u_1^{(2i-2,0)}\right)''(z),$$

$$u_1^{(0,2j)}(z) = \beta_2(z) \left(u_2^{(0,2j-2)}\right)''(z), \quad u_2^{(0,2j)}(z) = \beta_4(z) \left(u_2^{(0,2j-2)}\right)''(z),$$

and so

$$\left\|u_l^{(2i,0)}\right\|_{\infty,G_\delta} \leq \|B\|_{\infty,G} \left\| \left(u_1^{(2i-2,0)}\right)'' \right\|_{\infty,G_\delta}$$

and

$$\left\|u_l^{(0,2j)}\right\|_{\infty,G_\delta} \leq \|B\|_{\infty,G} \left\| \left(u_2^{(0,2j-2)}\right)'' \right\|_{\infty,G_\delta},$$

for $l = 1, 2$. We can use Lemma 2 of [9] directly on these equations to get the claim for i or $j = 0$, while for $i, j > 0$, we have

$$\left\|u_l^{(2i,2j)}\right\|_{\infty,G_\delta} \leq \|B\|_{\infty,G_\delta} \left[\left\| \left(u_1^{(2i-2,2j)}\right)'' \right\|_{\infty,G_\delta} + \left\| \left(u_2^{(2i,2j-2)}\right)'' \right\|_{\infty,G_\delta} \right], \quad l = 1, 2. \tag{28}$$

We proceed by induction on $2i + 2j$. Assume the result holds for $2i + 2j \leq m$ and look at $2i + 2j = m + 2$ (i.e. $2i + 2j - 2 = m$). Recall that $K^2 > 4e \|B\|_{\infty,G}$ and let $\kappa \in (0,1)$ be arbitrary.

We notice that the closed disc of radius $\kappa\delta$ and center z_0 is completely contained in $G_{(1-\kappa)\delta} \subseteq G$ for any such $z_0 \in G_\delta$. Also, for any x on the circle of radius $\kappa\delta$ and center z_0,

$$\left|u_l^{(2i,2j)}(x)\right| \le \left\|u_l^{(2i,2j)}\right\|_{\infty,G_{(1-\kappa)\delta}}, l = 1, 2.$$

Since $2i - 2 + 2j = m = 2i + 2j - 2$, we can use the induction hypothesis on $u_1^{(2i-2,2j)}$ and $u_2^{(2i,2j-2)}$ with $G_{(1-\kappa)\delta}$, and Cauchy's Integral Theorem to get

$$\left|\left(u_1^{(2i-2,2j)}\right)''(z_0)\right| \le \frac{2}{(\kappa\delta)^2}[(1-\kappa)\delta]^{-(2i+2j-2)}K^{2i+2j-2}(2i+2j-2)!\left\|\overrightarrow{u}^{(0,0)}\right\|_{\infty,G}$$

and

$$\left|\left(u_2^{(2i,2j-2)}\right)''(z_0)\right| \le \frac{2}{(\kappa\delta)^2}[(1-\kappa)\delta]^{-(2i+2j-2)}K^{2i+2j-2}(2i+2j-2)!\left\|\overrightarrow{u}^{(0,0)}\right\|_{\infty,G}.$$

This is true for any $z_0 \in G_\delta$, so by (28),

$$\begin{aligned}\left\|u_l^{(2i,2j)}\right\|_{\infty,G_\delta} &\le \|B\|_{\infty,G}\frac{4}{(\kappa\delta)^2}[(1-\kappa)\delta]^{-(2i+2j-2)}K^{2i+2j-2}(2i+2j-2)!\left\|\overrightarrow{u}^{(0,0)}\right\|_{\infty,G}\\ &= \mathcal{C}\delta^{-(2i+2j)}K^{2i+2j}(2i+2j)!\left\|\overrightarrow{u}^{(0,0)}\right\|_{\infty,G},\end{aligned}$$

where

$$\mathcal{C} := \frac{4\|B\|_{\infty,G}}{K^2\kappa^2(1-\kappa)^{2i+2j-2}}\left(\frac{1}{(2i+2j)(2i+2j-1)}\right).$$

It remains to show that $\mathcal{C}$ is bounded by a constant. Note that $2i + 2j - 2 = m$, and consider the expression $\frac{1}{\kappa^2(1-\kappa)^m}\frac{1}{(m+2)(m+1)}$. Let $\kappa = \frac{1}{m+2}$ and so $1 - \kappa = \frac{m+1}{m+2}$. Then we have

$$\begin{aligned}\frac{1}{\kappa^2(1-\kappa)^m}\frac{1}{(m+2)(m+1)} &= (m+2)^2\left(\frac{m+2}{m+1}\right)^m\frac{1}{(m+2)(m+1)}\\ &= \left(\frac{m+2}{m+1}\right)^{m+1} = \left(1+\frac{1}{m+1}\right)^{m+1} \le e.\end{aligned}$$

Thus $\mathcal{C} \le \dfrac{4\|B\|_{\infty,G}}{K^2}e \le 1$ by choice of K and the claim is proven. ∎

Lemma 3. *Let $\overrightarrow{u}^{(2i,2j)}$ be defined as in (21)–(23). Then there exist constants $C, K_1, K_2 > 0$ depending only on A and $\overrightarrow{f}$ such that*

$$\left\|\left(\overrightarrow{u}^{(2i,2j)}\right)^{(n)}\right\|_{\infty,\Omega} \le CK_1^{2i+2j}K_2^n(2i+2j)!n! \tag{29}$$

for any i, j, $n \in \mathbb{N}_0$.

Proof. Since the functions β_i, $i = 1,\ldots,4$ of $B = A^{-1}$ are analytic on $\overline{\Omega}$, there exists a neighborhood $G \subset \mathbb{C}$ of $\overline{\Omega}$ on which all β_i are holomorphic and bounded. Also, since the functions f_i, $i = 1, 2$ of $\overrightarrow{f}$ are analytic on $\overline{\Omega}$, we can also assume f_i are holomorphic on G. Thus, by Lemma 2,

$$\left\|\left(\overrightarrow{u}^{(2i,2j)}\right)^{(n)}\right\|_{\infty,\Omega} \le K_1^{2i+2j}K_2^n(2i+2j)!n!\left\|\overrightarrow{u}^{(0,0)}\right\|_{\infty,G}.$$

The fact that $\left\|\overrightarrow{u}^{(0,0)}\right\|_{\infty,G} \le C$ gives us the desired inequality. ∎

The next theorem bounds the derivatives of $\overrightarrow{w}$, independently of ε and μ.

Theorem 4. *There exist constants C, $\overline{K}_1$, $\overline{K}_2 \in \mathbb{R}^+$ depending only on $\overrightarrow{f}$ and A such that if $4M\mu\overline{K}_1 < 1$, $\overrightarrow{w}(x)$ given by* (20)*, satisfies*

$$\left\| \overrightarrow{w}^{(n)} \right\|_{\infty,\Omega} \leq C\overline{K}_2^n n! \; \forall \, n \in \mathbb{N}_0. \tag{30}$$

Proof. Recall that

$$\overrightarrow{w}(x) = \sum_{i=0}^{M} \sum_{j=0}^{M} \varepsilon^{2i} \mu^{2j} \overrightarrow{u}^{(2i,2j)}.$$

Setting $\overline{K}_2 = K_2$ in Lemma 3, we get

$$\begin{aligned}
\left\| \overrightarrow{w}^{(n)} \right\|_{\infty,\Omega} &\leq C\overline{K}_2^n n! \sum_{i=0}^{M} \sum_{j=0}^{M} \varepsilon^{2i} \mu^{2j} (2i+2j)! K_1^{2i+2j} \\
&\leq C\overline{K}_2^n n! \sum_{i=0}^{M} \sum_{j=0}^{M} \varepsilon^{2i} \mu^{2j} (2i+2j)^{2i+2j} K_1^{2i+2j} \\
&= C\overline{K}_2^n n! \sum_{i=0}^{M} \sum_{j=0}^{M} (\varepsilon^{2i} K_1^{2i} (2i+2j)^{2i})(\mu^{2j} K_1^{2j} (2i+2j)^{2j}) \\
&\leq C\overline{K}_2^n n! \sum_{i=0}^{M} \sum_{j=0}^{M} (\varepsilon K_1 4M)^{2i} (\mu K_1 4M)^{2j}.
\end{aligned}$$

Since $4M\varepsilon K_1 < 4M\mu K_1 \leq q < 1$ for some q, the sums above can be bounded by a converging geometric series and we get

$$\left\| \overrightarrow{w}^{(n)} \right\|_{\infty,\Omega} \leq C\overline{K}_2^n n!.$$

Setting $\overline{K}_1 = \frac{K_1}{q}$ gives the desired bound. ☐

Remark 1. *By the previous result, we see that A^+ and A^- in the decomposition* (18) *for $\overrightarrow{u}$ are bounded independently of ε and μ.*

We now derive bounds on the boundary layer part $\overrightarrow{u}^-$. The bounds for $\overrightarrow{u}^+$ can be derived in an analogous way.

Theorem 5. *Let $\overrightarrow{u}^-$ be the solution of* (25)*. Then there exist constants C, $K > 0$ independent of ε, μ and n such that for any $x \in \overline{\Omega}$, $n \in \mathbb{N}_0$,*

$$\left| \left(u_1^-\right)^{(n)}(x) \right| \leq CK^n \left(e^{-x\alpha/\varepsilon} \max\{n, \varepsilon^{-1}\}^n + e^{-x\alpha/\mu} \max\{n, \mu^{-1}\}^n \right), \tag{31}$$

$$\left| \left(u_2^-\right)^{(n)}(x) \right| \leq CK^n \left(e^{-x\alpha/\varepsilon} \max\{n^n, \mu^{-2}\varepsilon^{-n+2}\} + e^{-x\alpha/\mu} \max\{n, \mu^{-1}\}^n \right). \tag{32}$$

Proof. We will first prove (32), by induction on n. The cases $n=0$ and $n=1$ were shown in [6], so assume (32) holds for $0\le k\le n+1$ and establish it for $n+2$. We have from (1)

$$\mu^2\left(u_2^-\right)''(x)=a_{21}(x)u_1^-(x)+a_{22}(x)u_2^-(x).$$

Using (7) and the induction hypothesis, we obtain

$$\begin{aligned}
\mu^2\left|\left(u_2^-\right)^{(n+2)}(x)\right| &\le \sum_{k=0}^{n}\binom{n}{k}\left[\left|a_{21}^{(k)}(x)\right|\,\left|\left(u_1^-\right)^{(n-k)}(x)\right|+\left|a_{22}^{(k)}(x)\right|\,\left|\left(u_2^-\right)^{(n-k)}(x)\right|\right]\\
&\le \sum_{k=0}^{n}\binom{n}{k}\Big[2C_a\gamma_a^k k!CK^{n-k}\Big\{e^{-x\alpha/\varepsilon}\max\{(n-k)^{n-k},\mu^{-2}\varepsilon^{-n+k+2}\}+\\
&\quad +e^{-x\alpha/\mu}\max\{n-k,\mu^{-1}\}^{n-k}\Big\}\Big]\\
&\le 2CK^nC_a\sum_{k=0}^{n}\frac{n!}{(n-k)!}\left(\frac{\gamma_a}{K}\right)^k\left[e^{-x\alpha/\varepsilon}\max\{n,\varepsilon^{-1}\}^n+e^{-x\alpha/\mu}\max\{n,\mu^{-1}\}^n\right]\\
&\le 2CK^{n+2}\left[\frac{C_aK^{-2}}{1-\frac{\gamma_a}{K}}\right]\left[e^{-x\alpha/\varepsilon}\max\{n,\varepsilon^{-1}\}^n+e^{-x\alpha/\mu}\max\{n,\mu^{-1}\}^n\right].
\end{aligned}$$

We may choose $K>\max\{1,C_a,\gamma_a\}$ so that the fraction in brackets above is bounded by 1 hence

$$\begin{aligned}
\left|\left(u_2^-\right)^{(n+2)}(x)\right| &\le 2CK^{n+2}\left(e^{-x\alpha/\varepsilon}\mu^{-2}\max\{n+2,\varepsilon^{-1}\}^n+e^{-x\alpha/\mu}\mu^{-2}\max\{n+2,\mu^{-1}\}^n\right)\\
&\le CK^{n+2}\left(e^{-x\alpha/\varepsilon}\max\{(n+2)^{n+2},\mu^{-2}\varepsilon^{-n}\}+e^{-x\alpha/\mu}\max\{n+2,\mu^{-1}\}^{n+2}\right),
\end{aligned}$$

for all $x\in\overline{\Omega}$ as desired. We note that, since $\varepsilon^{-2}>\mu^{-2}$, the above expression can also be written as

$$\left|\left(u_2^-\right)^{(n+2)}(x)\right|\le CK^{n+2}\left(e^{-x\alpha/\varepsilon}\max\{n+2,\varepsilon^{-1}\}^{n+2}+e^{-x\alpha/\mu}\max\{n+2,\mu^{-1}\}^{n+2}\right). \tag{33}$$

Repeating the previous argument for u_1^- yields

$$\left|\left(u_1^-\right)^n(x)\right|\le CK^n\max\{n,\varepsilon^{-1}\}^n\left[e^{-x\alpha/\varepsilon}+e^{-x\alpha/\mu}\right]\quad\forall x\in\overline{\Omega}$$

which is not quite what we want. So, we proceed in a different manner. Define the scalar operator

$$L_\varepsilon u:=-\varepsilon^2u''+a_{11}u$$

and note that

$$\left|L_\varepsilon\left(u_1^-\right)^{(n)}(x)\right|=\left|-\varepsilon^2\left(u_1^-\right)^{(n+2)}(x)+a_{11}(x)\left(u_1^-\right)^{(n)}(x)\right|.$$

The first component of $L(\overrightarrow{u}^-)=\overrightarrow{0}$ implies

$$\begin{aligned}
-\varepsilon^2\left(u_1^-\right)^{(n+2)}(x) &= \Big(-a_{11}(x)u_1^-(x)-a_{12}(x)u_2^-(x)\Big)^{(n)}\\
&= -\sum_{\nu=0}^{n}\binom{n}{k}\left(a_{11}^{(\nu)}(x)\left(u_1^-\right)^{(n-\nu)}(x)+a_{12}^{(\nu)}(x)\left(u_2^-\right)^{(n-\nu)}(x)\right).
\end{aligned}$$

Using (33), we have

$$
\begin{aligned}
\left|L_\varepsilon\left(u_1^-\right)^{(n)}(x)\right| &= \left|a_{11}(x)\left(u_1^-\right)^{(n)}(x) - \sum_{\nu=0}^{n}\binom{n}{\nu}\left(a_{11}^{(\nu)}(x)\left(u_1^-\right)^{(n-\nu)}(x) + a_{12}^{(\nu)}(x)\left(u_2^-\right)^{(n-\nu)}(x)\right)\right| \\
&= \left|-a_{12}(x)\left(u_2^-\right)^{(n)}(x) - \sum_{\nu=1}^{n}\binom{n}{\nu}\left(a_{11}^{(\nu)}(x)\left(u_1^-\right)^{(n-\nu)}(x) + a_{12}^{(\nu)}(x)\left(u_2^-\right)^{(n-\nu)}(x)\right)\right| \\
&\leq \left|a_{12}(x)\right|\left|\left(u_2^-\right)^{(n)}(x)\right| + \sum_{\nu=1}^{n}\binom{n}{\nu}\Bigg(\left|a_{11}^{(\nu)}(x)\right|\left|\left(u_1^-\right)^{(n-\nu)}(x)\right| + \\
&\qquad \left|a_{12}^{(\nu)}(x)\right|\left|\left(u_2^-\right)^{(n-\nu)}(x)\right|\Bigg) \\
&\leq C_a C K^n\left(e^{-x\alpha/\varepsilon}\max\{n,\varepsilon^{-1}\}^n + e^{-x\alpha/\mu}\max\{n,\mu^{-1}\}^n\right) \\
&\quad + \sum_{\nu=1}^{n}\frac{n!}{\nu!(n-\nu)!}\left\{2C_a\gamma_1^\nu \nu!\left(\left|\left(u_1^-\right)^{(n-\nu)}(x)\right| + \left|\left(u_2^-\right)^{(n-\nu)}(x)\right|\right)\right\}.
\end{aligned}
$$

Using $\dfrac{n!}{(n-\nu)!} \leq n^\nu$ and the induction hypothesis we get

$$
\begin{aligned}
\left|L_\varepsilon\left(u_1^-\right)^{(n)}(x)\right| &\leq C_a C K^n\left(e^{-x\alpha/\varepsilon}\max\{n,\varepsilon^{-1}\}^n + e^{-x\alpha/\mu}\max\{n,\mu^{-1}\}^n\right) \\
&\quad + 2C_a C K^n\sum_{\nu=1}^{n} n^\nu\left(\frac{\gamma_a}{K}\right)^\nu\Big(e^{-x\alpha/\varepsilon}\max\{n-\nu,\varepsilon^{-1}\}^{n-\nu} \\
&\qquad + e^{-x\alpha/\mu}\max\{n-\nu,\mu^{-1}\}^{n-\nu}\Big) \\
&\leq C_a C K^n\left(e^{-x\alpha/\varepsilon}\max\{n,\varepsilon^{-1}\}^n + e^{-x\alpha/\mu}\max\{n,\mu^{-1}\}^n\right) \\
&\quad + 2C_a C K^n\sum_{\nu=1}^{n}\left(\frac{\gamma_a}{K}\right)^\nu\left(e^{-x\alpha/\varepsilon}\max\{n,\varepsilon^{-1}\}^n + e^{-x\alpha/\mu}\max\{n,\mu^{-1}\}^n\right).
\end{aligned}
$$

Since $K > \max\{1, C_a, \gamma_a\}$, we may bound the finite sum above with a converging geometric series and obtain

$$
\left|L_\varepsilon\left(u_1^-\right)^{(n)}(x)\right| \leq \overline{C}K^n\left(e^{-x\alpha/\varepsilon}\max\{n,\varepsilon^{-1}\}^n + e^{-x\alpha/\mu}\max\{n,\mu^{-1}\}^n\right) \quad \forall x \in \Omega.
$$

We have already seen that on $\partial\Omega$

$$
\left|\left(u_1^-\right)^{(n)}(x)\right| \leq C K^n\max\{n,\varepsilon^{-1}\}^n\left[e^{-x\alpha/\varepsilon} + e^{-x\alpha/\mu}\right].
$$

Since $e^{-x\alpha/\varepsilon} + e^{-x\alpha/\mu} \leq 2$ for all $x \in \partial\Omega$, we further have

$$
\left|\left(u_1^-\right)^{(n)}(x)\right| \leq 2CK^n\max\{n,\varepsilon^{-1}\}^n
$$

for all $x \in \partial\Omega$. We now define the barrier function

$$
\Psi(x) = \widehat{C}K^n\left(e^{-x\alpha/\varepsilon}\max\{n,\varepsilon^{-1}\}^n + e^{-x\alpha/\mu}\max\{n,\mu^{-1}\}^n\right)
$$

with $\widehat{C}$ sufficiently large so that $\Psi(x) \geq \left|\left(u_1^-\right)^{(n)}(x)\right|$ on $\partial\Omega$ and

$$
\widehat{C}(a_{11}(x) - \alpha^2) \geq \overline{C} \quad \forall\, x \in \Omega.
$$

Recall that $\varepsilon < \mu$ and so $a_{11}(x) - \dfrac{\varepsilon^2}{\mu^2}\alpha^2 \geq a_{11}(x) - \alpha^2$. Then, for any $x \in \Omega$,

$$\begin{aligned}
L_\varepsilon \Psi(x) &= \varepsilon^2 \Psi''(x) + a_{11}(x)\Psi(x) \\
&= \widehat{C}K^n \Big\{ -\varepsilon^2 \Big((e^{-x\alpha/\varepsilon})'' \max\{n, \varepsilon^{-1}\}^n + (e^{-x\alpha/\mu})'' \max\{n, \mu^{-1}\} \Big) \\
&\qquad + a_{11}(x) \Big(e^{-x\alpha/\varepsilon} \max\{n, \varepsilon^{-1}\}^n + e^{-x\alpha/\mu} \max\{n, \mu^{-1}\}^n \Big) \Big\} \\
&= \widehat{C}K^n \left\{ -\varepsilon^2 \Big(\frac{\alpha^2}{\varepsilon^2} e^{-x\alpha/\varepsilon} \max\{n, \varepsilon^{-1}\}^n + \frac{\alpha^2}{\mu^2} e^{-x\alpha/\mu} \max\{n, \mu^{-1}\}^n \Big) \right. \\
&\qquad \left. + a_{11}(x) \left(e^{-x\alpha/\varepsilon} \max\{n, \varepsilon^{-1}\}^n + e^{-x\alpha/\mu} \max\{n, \mu^{-1}\}^n \right) \right\} \\
&= \widehat{C}K^n \left\{ e^{-x\alpha/\varepsilon} \max\{n, \varepsilon^{-1}\}^n (a_{11}(x) - \alpha^2) + e^{-x\alpha/\mu} \max\{n, \mu^{-1}\}^n \left(a_{11} - \frac{\varepsilon^2}{\mu^2}\alpha^2 \right) \right\} \\
&\geq K^n \widehat{C}(a_{11} - \alpha^2) \Big(e^{-x\alpha/\varepsilon} \max\{n, \varepsilon^{-1}\}^n + e^{-x\alpha/\mu} \max\{n, \mu^{-1}\}^n \Big) \\
&\geq K^n \overline{C} \Big(e^{-x\alpha/\varepsilon} \max\{n, \varepsilon^{-1}\}^n + e^{-x\alpha/\mu} \max\{n, \mu^{-1}\}^n \Big).
\end{aligned}$$

By the Comparison Principle (Lemma 1 of [6]), we get that $\Psi(x) \geq \left| \left(u_1^-\right)^{(n)}(x) \right|$ for all $x \in \overline{\Omega}$; i.e.,

$$\left| \left(u_1^-\right)^{(n)}(x) \right| \leq \widehat{C}K^n \left(e^{-x\alpha/\varepsilon} \max\{n, \varepsilon^{-1}\}^n + e^{-x\alpha/\mu} \max\{n, \mu^{-1}\}^n \right)$$

and this completes the proof. □

The following result establishes that the boundary layer functions can be separated into a part that depends on ε and a part that depends on μ. This was also shown in [6], where derivative growth estimates were established for the first three derivatives of $\vec{u}^-$. Here, we extend those results and obtain estimates valid for a much higher number of derivatives, something that will be needed in Section 3 ahead.

Lemma 6. *There exist functions $u_{1,\varepsilon}^-$, $u_{1,\mu}^-$, $u_{2,\varepsilon}^-$ and $u_{2,\mu}^-$ such that*

$$u_1^-(x) = u_{1,\varepsilon}^-(x) + u_{1,\mu}^-(x), \qquad u_2^-(x) = u_{2,\varepsilon}^-(x) + u_{2,\mu}^-(x).$$

In addition, there exist constants C, $K > 0$, independent of ε and μ such that for $n = 0, 1, 2, ..., q \in \mathbb{N}$, for some $q < \alpha\mu^{-1}$,

$$\left| \left(u_{1,\varepsilon}^-\right)^{(n)}(x) \right| \leq CK^n e^{-x\alpha/\varepsilon} \max\left\{n, \varepsilon^{-1}\right\}^n \tag{34}$$

$$\left| \left(u_{2,\varepsilon}^-\right)^{(n)}(x) \right| \leq CK^n e^{-x\alpha/\varepsilon} \max\left\{n^n, \mu^{-2}\varepsilon^{-n+2}\right\} \tag{35}$$

$$\left| \left(u_{i,\mu}^-\right)^{(n)}(x) \right| \leq CK^n e^{-x\alpha/\mu} \max\left\{n, \mu^{-1}\right\}^n, \ i = 1, 2 \tag{36}$$

for all $x \in \overline{\Omega}$.

Proof. We note that the above result was established in [6] for $n = 0, 1, 2$, so we prove it for $n > 2$. Concentrating first on $u_{1,\varepsilon}^-(x), u_{1,\mu}^-(x)$, we define

$$x^* = \frac{\varepsilon\mu n \ln(\mu/\varepsilon)}{\alpha(\mu-\varepsilon)} > 0,$$

and we have

$$\varepsilon^{-n} e^{-x^*\alpha/\varepsilon} = \mu^{-n} e^{-x^*\alpha/\mu}.$$

Now, there exists $\zeta^* \in (\varepsilon, \mu)$ such that

$$x^* = \frac{\varepsilon\mu n \left(\ln(\mu) - \ln(\varepsilon)\right)}{\alpha(\mu-\varepsilon)} = \frac{\varepsilon\mu n}{\alpha}\frac{1}{\zeta^*} < \frac{\varepsilon\mu n}{\alpha}\frac{1}{\varepsilon} = \frac{\mu n}{\alpha},$$

so if $n < \alpha/\mu$ we have $x^* \in (0,1)$. Moreover, observe that on $[0, x^*)$ we have $\varepsilon^{-n}e^{-x\alpha/\varepsilon} > \mu^{-n}e^{-x\alpha/\mu}$, while on $(x^*, 1]$ we have the opposite. Now, define $u_{1,\mu}^-(x)$ on $[0,1]$ by

$$u_{1,\mu}^-(x) = \begin{cases} \sum_{i=0}^{n} \frac{(x-x^*)^i}{i!} \left(u_1^-\right)^{(i)}(x^*) & , x \in [0, x^*) \\ u_1^-(x) & , x \in (x^*, 1] \end{cases}$$

and set

$$u_{1,\varepsilon}^-(x) = u_1^-(x) - u_{1,\mu}^-(x).$$

By construction, $u_{1,\varepsilon}^-(x), u_{1,\mu}^-(x) \in C^n([0,1])$, and on $[0, x^*]$ we have

$$\begin{aligned} \left|\left(u_{1,\mu}^-\right)^{(n)}(x)\right| &= \left|\left(u_1^-\right)^{(n)}(x^*)\right| \le CK^n\left(e^{-x^*\alpha/\varepsilon}\max\{n,\varepsilon^{-1}\}^n + e^{-x^*\alpha/\mu}\max\{n,\mu^{-1}\}^n\right) \\ &\le 2CK^n e^{-x^*\alpha/\mu}\max\{n,\mu^{-1}\}^n \le \overline{C}K^n e^{-x\alpha/\mu}\max\{n,\mu^{-1}\}^n. \end{aligned}$$

On $(x^*, 1]$ we have

$$\begin{aligned} \left|\left(u_{1,\mu}^-\right)^{(n)}(x)\right| &= \left|\left(u_1^-\right)^{(n)}(x)\right| \le CK^n\left(e^{-x\alpha/\varepsilon}\max\{n,\varepsilon^{-1}\}^n + e^{-x\alpha/\mu}\max\{n,\mu^{-1}\}^n\right) \\ &\le CK^n e^{-x\alpha/\mu}\max\{n,\mu^{-1}\}^n, \end{aligned}$$

since for $x \in (x^*, 1]$ we have $\varepsilon^{-n}e^{-x\alpha/\varepsilon} < \mu^{-n}e^{-x\alpha/\mu}$. Thus,

$$\left|\left(u_{1,\mu}^-\right)^{(n)}(x)\right| \le CK^n e^{-x\alpha/\mu}\max\{n,\mu^{-1}\}^n \ , \ \forall\, x \in \overline{\Omega}.$$

Next, notice that $u_{1,\varepsilon}^-(x) = 0 \ \forall\ x \in [x^*, 1]$, and for $x \in [0, x^*)$ we have $\varepsilon^{-n}e^{-x\alpha/\varepsilon} > \mu^{-n}e^{-x\alpha/\mu}$, which gives

$$\begin{aligned} \left|\left(u_{1,\varepsilon}^-\right)^{(n)}(x)\right| &\le \left|\left(u_1^-\right)^{(n)}(x)\right| + \left|\left(u_{1,\mu}^-\right)^{(n)}(x)\right| \\ &\le CK^n\left(e^{-x\alpha/\varepsilon}\max\{n,\varepsilon^{-1}\}^n + e^{-x\alpha/\mu}\max\{n,\mu^{-1}\}^n\right) \\ &\le \overline{C}K^n e^{-x\alpha/\varepsilon}\max\{n,\varepsilon^{-1}\}^n \ , \ \forall\, x \in \overline{\Omega}. \end{aligned}$$

To establish the desired bounds for $u^-_{2,\varepsilon}(x), u^-_{2,\mu}(x)$ we proceed in a similar fashion by defining

$$\widehat{x} = \frac{\varepsilon\mu\,(n-2)\ln(\mu/\varepsilon)}{\alpha(\mu-\varepsilon)} > 0$$

and noting that

$$\mu^{-2}\varepsilon^{-n+2}e^{-\widehat{x}\alpha/\varepsilon} = \mu^{-n}e^{-\widehat{x}\alpha/\mu}.$$

Thus, if $n-2 < \alpha/\mu$ then $\widehat{x} \in (0,1)$ and on $[0,\widehat{x})$ we have $\mu^{-2}\varepsilon^{-n+2}e^{-x\alpha/\varepsilon} > \mu^{-n}e^{-x\alpha/\mu}$, while on $(\widehat{x},1]$ we have the opposite. Define $u^-_{2,\mu}(x)$ on $[0,1]$ by

$$u^-_{2,\mu}(x) = \begin{cases} \sum\limits_{i=0}^{n} \frac{(x-\widehat{x})^i}{i!}\left(u^-_2\right)^{(i)}(\widehat{x}) & , x \in [0,\widehat{x}) \\ u^-_2(x) & , x \in (\widehat{x},1] \end{cases}$$

and set

$$u^-_{2,\varepsilon}(x) = u^-_2(x) - u^-_{2,\mu}(x).$$

Proceeding as above, we see that on $[0,\widehat{x})$

$$\begin{aligned} \left|\left(u^-_{2,\mu}\right)^{(n)}(x)\right| &= \left|\left(u^-_2\right)^{(n)}(\widehat{x})\right| \leq CK^n\left(e^{-\widehat{x}\alpha/\varepsilon}\max\{n^n,\mu^{-2}\varepsilon^{-n+2}\} + e^{-\widehat{x}\alpha/\mu}\max\{n,\mu^{-1}\}^n\right) \\ &\leq 2CK^ne^{-\widehat{x}\alpha/\mu}\max\{n,\mu^{-1}\}^n \leq \widehat{C}K^ne^{-x\alpha/\mu}\max\{n,\mu^{-1}\}^n. \end{aligned}$$

On $(\widehat{x},1]$ we have

$$\begin{aligned} \left|\left(u^-_{2,\mu}\right)^{(n)}(x)\right| &= \left|\left(u^-_2\right)^{(n)}(x)\right| \leq CK^n\left(e^{-x\alpha/\varepsilon}\max\{n^n,\mu^{-2}\varepsilon^{-n+2}\} + e^{-x\alpha/\mu}\max\{n,\mu^{-1}\}^n\right) \\ &\leq CK^ne^{-x\alpha/\mu}\max\{n,\mu^{-1}\}^n, \end{aligned}$$

hence

$$\left|\left(u^-_{2,\mu}\right)^{(n)}(x)\right| \leq CK^ne^{-x\alpha/\mu}\max\{n,\mu^{-1}\}^n\ ,\ \forall\, x \in \overline{\Omega}.$$

It remains to establish the bound for $u^-_{2,\varepsilon}$, which satisfies $u^-_{2,\varepsilon}(x) = 0\ \forall\, x \in [\widehat{x},1]$. For $x \in [0,\widehat{x})$ we have

$$\begin{aligned} \left|\left(u^-_{2,\varepsilon}\right)^{(n)}(x)\right| &\leq \left|\left(u^-_2\right)^{(n)}(x)\right| + \left|\left(u^-_{2,\mu}\right)^{(n)}(x)\right| \\ &\leq CK^n\left(e^{-x\alpha/\varepsilon}\max\{n^n,\mu^{-2}\varepsilon^{-n+2}\} + e^{-x\alpha/\mu}\max\{n,\mu^{-1}\}^n\right) \\ &\leq \widehat{C}K^ne^{-x\alpha/\varepsilon}\max\{n^n,\mu^{-2}\varepsilon^{-n+2}\}\ ,\ \forall\, x \in \overline{\Omega} \end{aligned}$$

and the proof is complete. □

The final theorem of this section gives bounds on the remainder $\overrightarrow{r}$ in terms of μ, the order M of the asymptotic expansion (18) and the input data.

Theorem 7. *There exists constants $C, K_1 > 0$ depending only on the input data A and $\overrightarrow{f}$ such that if $4M\mu K_1 < 1$, the remainder $\overrightarrow{r}$ defined by (26) satisfies*

$$\|\overrightarrow{r}\|_{E,\Omega} \leq C\mu^2\left(4M\mu K_1\right)^{2M}. \tag{37}$$

Proof. Recall that $\overrightarrow{r}$ satisfies (26), thus the a priori estimate (14) gives

$$\|\overrightarrow{r}\|_{E,\Omega} \leq \frac{1}{\alpha}\left\{\varepsilon^{2M+2}\sum_{j=0}^{M}\mu^{2j}\left\|\left(u_1^{(2M,2j)}\right)''\right\|_{0,\Omega} + \mu^{2M+2}\sum_{i=0}^{M}\varepsilon^{2i}\left\|\left(u_2^{(2i,2M)}\right)''\right\|_{0,\Omega}\right\}.$$

By Lemma 3, and the fact that $(a+b)! \leq 2\,(a+b)^{a+b}\ \forall\, a,b \geq 0$, we get

$$\begin{aligned}
\|\overrightarrow{r}\|_{E,\Omega} &\leq \frac{1}{\alpha}\left\{\varepsilon^{2M+2}\sum_{j=0}^{M}\mu^{2j}K_1^{2M+2j}K_2^2\,(2M+2j)!2! + \mu^{2M+2}\sum_{i=0}^{M}\varepsilon^{2i}K_1^{2i+2M}K_2^2\,(2i+2M)!2!\right\} \\
&\leq \frac{2}{\alpha}\mu^{2M+2}K_1^{2M}K_2^2\sum_{j=0}^{M}\mu^{2j}K_1^{2j}\,(2M+2j)! \\
&\leq \frac{2}{\alpha}\mu^{2M+2}K_1^{2M}K_2^2\sum_{j=0}^{M}\mu^{2j}K_1^{2j}\,(2M+2j)^{(2M+2j)} \\
&\leq \frac{2}{\alpha}\mu^{2M+2}K_1^{2M}K_2^2\sum_{j=0}^{M}\mu^{2j}K_1^{2j}\,(4M)^{(2M+2j)} \\
&\leq C\mu^2\,(4M\mu K_1)^{2M}\sum_{j=0}^{M}(4M\mu K_1)^{2j}.
\end{aligned}$$

Since $4M\mu K_1 < 1$, the above series can be bounded by a converging geometric series, and we have the result. □

Remark 2. *Theorem 7 shows that the remainder is small provided $4M\mu$ is small. In the case when $4M\mu$ is large the asymptotic expansion is not meaningful.*

3 The Finite Element Method

In this section we describe the specific choice of the subspace S_N, which will allow us to approximate the solution of (15) at an exponential rate.

Let $\Delta = \{0 = x_0 < x_1 < ... < x_{\mathcal{M}} = 1\}$ be an arbitrary partition of $\Omega = (0,1)$ and set

$$I_j = (x_{j-1}, x_j)\,,\ h_j = x_j - x_{j-1},\ j = 1, ..., \mathcal{M}.$$

Also, define the master (or standard) element $I_{ST} = (-1,1)$, and note that it can be mapped onto the j^{th} element I_j by the linear mapping

$$x = Q_j(t) = \frac{1}{2}(1-t)\,x_{j-1} + \frac{1}{2}(1+t)\,x_j.$$

With $\Pi_p(I_{ST})$ the space of polynomials of degree $\leq p$ on I_{ST}, we define our finite dimensional subspaces as

$$S_N \equiv S^{\overrightarrow{p}}(\Delta) = \left\{u \in H_0^1(\Omega) : u\,(Q_j(t)) \in \Pi_{p_j}(I_{ST})\,, j = 1, ..., \mathcal{M}\right\}$$

and

$$\overrightarrow{S}_0^p(\Delta) := [S^{\overrightarrow{p}}(\Delta) \cap H_0^1(\Omega)]^2, \tag{38}$$

where $\overrightarrow{p} = (p_1, ..., p_{\mathcal{M}})$ is the vector of polynomial degrees assigned to the elements.

The following approximation result from [15] will be the main tool for the analysis of the method.

Theorem 8. *For any $u \in C^\infty\left(\overline{I}_{ST}\right)$ there exists $\mathcal{I}_p u \in \Pi_p\left(I_{ST}\right)$ such that*

$$u\left(\pm 1\right) = \mathcal{I}_p u\left(\pm 1\right), \tag{39}$$

$$\left\| u - \mathcal{I}_p u \right\|_{0,I_{ST}}^2 \leq \frac{1}{p^2} \frac{(p-s)!}{(p+s)!} \left\| u^{(s+1)} \right\|_{0,I_{ST}}^2, \ \forall\, s = 0, 1, ..., p, \tag{40}$$

$$\left\| (u - \mathcal{I}_p u)' \right\|_{0,I_{ST}}^2 \leq \frac{(p-s)!}{(p+s)!} \left\| u^{(s+1)} \right\|_{0,I_{ST}}^2, \ \forall\, s = 0, 1, ..., p. \tag{41}$$

The definition below describes the mesh used for the method: If we are in the asymptotic range of p, i.e. $p \geq 1/\varepsilon > 1/\mu$, then a single element suffices since p will be sufficiently large to give us exponential convergence without any refinement. If we are in the pre-asymptotic range, i.e. $p < 1/\mu < 1/\varepsilon$, then the mesh consists of five elements as described below. We should point out that this is the *minimal* mesh-degree combination for attaining exponential convergence; obviously, refining within each element will retain the convergence rate but would require more degrees of freedom – one such example is the so-called *geometrically graded* mesh discussed in [9] for the scalar problem.

Definition 9. *For $\kappa > 0$, $p \in \mathbb{N}$ and $0 < \varepsilon < \mu << 1$, define the spaces $\overrightarrow{S}(\kappa, p)$ of piecewise polynomials by*

$$\overrightarrow{S}(\kappa,p) := \begin{cases} \overrightarrow{S}_0^p(\Delta);\ \Delta = \{0,1\} & \text{if } \kappa p\varepsilon \geq \frac{1}{2} \\ \overrightarrow{S}_0^p(\Delta);\ \Delta = \{0, \kappa p\varepsilon, \kappa p\mu, 1-\kappa p\mu, 1-\kappa p\varepsilon, 1\} & \text{if } \kappa p\mu < \frac{1}{2} \\ \overrightarrow{S}^p(\Delta);\ \Delta = \{0, \kappa p\varepsilon, 1-\kappa p\varepsilon, 1\} & \text{if } \kappa p\varepsilon < \frac{1}{2}, \text{ and } \kappa p\mu \geq \frac{1}{2}. \end{cases}$$

In all cases above the polynomial degree is uniformly p on all elements.

Before we state the main theorem of the paper, we present a useful computation.

Lemma 10. *Let $p \in \mathbb{N}, \lambda \in (0,1]$. Then*

$$\frac{(p-\lambda p)!}{(p+\lambda p)!} \leq \left[\frac{(1-\lambda)^{(1-\lambda)}}{(1+\lambda)^{(1+\lambda)}}\right]^p p^{-2\lambda p} e^{2\lambda p+1}.$$

Proof. Using Stirling's approximation

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n+1}} \leq n! \leq \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n}} \leq \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e$$

for the factorial (cf. [13]), we have

$$\begin{aligned}\frac{(p-\lambda p)!}{(p+\lambda p)!} &\leq \frac{\sqrt{2\pi(1-\lambda)p}}{\sqrt{2\pi(1+\lambda)p}}\frac{\left(\frac{(1-\lambda)p}{e}\right)^{(1-\lambda)p}}{\left(\frac{(1+\lambda)p}{e}\right)^{(1+\lambda)p}}\frac{e}{e^{\frac{1}{12(1+\lambda)p+1}}} \leq \frac{[(1-\lambda)\,p]^{(1-\lambda)p}}{[(1+\lambda)\,p]^{(1+\lambda)p}}e^{2\lambda p}e^{1-\frac{1}{12(1+\lambda)p+1}} \\ &\leq \left[\frac{(1-\lambda)^{(1-\lambda)}}{(1+\lambda)^{(1+\lambda)}}\right]^p p^{-2\lambda p}e^{2\lambda p}e.\end{aligned}$$

□

We now present our main result.

Theorem 11. *Let $\overrightarrow{f}$ and A be composed of functions that are analytic on $\overline{\Omega}$ and satisfy the conditions in (4)–(7). Let $\overrightarrow{u} = [u_1, u_2]^T$ be the solution to (1), (3). Then there exist constants $\kappa, C, \beta > 0$ depending only on $\overrightarrow{f}$ and A such that there exists $\mathcal{I}_p\overrightarrow{u} = [\mathcal{I}_p u_1, \mathcal{I}_p u_2]^T \in \overrightarrow{S}(\kappa, p)$ with $\mathcal{I}_p\overrightarrow{u} = \overrightarrow{u}$ on $\partial\Omega$ and*

$$\|\overrightarrow{u} - \mathcal{I}_p\overrightarrow{u}\|_{E,\Omega}^2 \leq Cp^3e^{-\beta p}.$$

Proof.
Case 1. $\kappa p\varepsilon \geq \frac{1}{2}$, i.e. $p \geq \frac{1}{2\kappa\varepsilon}$ (asymptotic case), $\Delta = \{0, 1\}$

From Theorem 1 we have

$$\left\|\overrightarrow{u}^{(n)}\right\|_{0,\Omega}^2 \leq CK^{2n}\max\{n, \varepsilon^{-1}\}^{2n} \quad \forall n \in \mathbb{N}_0,$$

and by Theorem 8 there exists $\mathcal{I}_p\overrightarrow{u} \in \overrightarrow{S}(\kappa, p)$ such that $\overrightarrow{u} = \mathcal{I}_p\overrightarrow{u}$ on $\partial\Omega$ and for $0 \leq s \leq p$

$$\|(\overrightarrow{u} - \mathcal{I}_p\overrightarrow{u})'\|_{0,\Omega}^2 \leq \frac{(p-s)!}{(p+s)!}\left\|\overrightarrow{u}^{(s+1)}\right\|_{0,\Omega}^2 \leq \frac{(p-s)!}{(p+s)!}CK^{2(s+1)}\max\{s+1, \varepsilon^{-1}\}^{2(s+1)}.$$

Choose $s = \lambda p$, for some $\lambda \in (0, 1]$. Then, since $p \geq 1/(2\kappa\varepsilon)$, we have

$$\max\{s+1, \varepsilon^{-1}\}^{2(s+1)} = \max\{\lambda p+1, \varepsilon^{-1}\}^{2(\lambda p+1)} = (\lambda p+1)^{2(\lambda p+1)},$$

which, along with Lemma 10, gives

$$\begin{aligned}\|(\overrightarrow{u} - \mathcal{I}_p\overrightarrow{u})'\|_{0,\Omega}^2 &\leq \frac{(p-\lambda p)!}{(p+\lambda p)!}CK^{2(\lambda p+1)}(\lambda p+1)^{2(\lambda p+1)} \\ &\leq \left[\frac{(1-\lambda)^{(1-\lambda)}}{(1+\lambda)^{(1+\lambda)}}\right]^p p^{-2\lambda p}e^{2\lambda p+1}CK^{2(\lambda p+1)}(\lambda p+1)^{2(\lambda p+1)} \\ &\leq CeK^2\left[\frac{(1-\lambda)^{(1-\lambda)}}{(1+\lambda)^{(1+\lambda)}}(eK)^{2\lambda}\right]^p(\lambda p+1)^2\left(\frac{1+\lambda p}{p}\right)^{2\lambda p} \\ &\leq CeK^2p^2\left[\frac{(1-\lambda)^{(1-\lambda)}}{(1+\lambda)^{(1+\lambda)}}(eK)^{2\lambda}\right]^p\left(\frac{1}{p}+\lambda\right)^{2\lambda p}.\end{aligned}$$

Since $\left(\frac{1}{p}+\lambda\right)^{2\lambda p} = \lambda^{2\lambda p}\left[\left(1+\frac{1}{\lambda p}\right)^{\lambda p}\right]^2 \leq e^2\lambda^{2\lambda p}$, we further get

$$\|(\overrightarrow{u}-\mathcal{I}_p\overrightarrow{u})'\|_{0,\Omega}^2 \leq Cp^2\left[\frac{(1-\lambda)^{(1-\lambda)}}{(1+\lambda)^{(1+\lambda)}}(eK\lambda)^{2\lambda}\right]^p,$$

so if we choose $\lambda = (eK)^{-1} \in (0,1)$ we have

$$\|(\overrightarrow{u}-\mathcal{I}_p\overrightarrow{u})'\|_{0,\Omega}^2 \leq Cp^2e^{-bp}, \tag{42}$$

where $b = |\ln q|$, $q = \frac{(1-\lambda)^{(1-\lambda)}}{(1+\lambda)^{(1+\lambda)}} < 1$, and the constant $C > 0$ is independent of ε and μ. Repeating the previous argument for the L^2 norm of $(\overrightarrow{u}-\mathcal{I}_p\overrightarrow{u})$, we get, using (41),

$$\|\overrightarrow{u}-\mathcal{I}_p\overrightarrow{u}\|_{0,\Omega}^2 \leq Ce^{-bp}, \tag{43}$$

with $C > 0$ independent of ε and μ. Combining (42)–(43), and using the definition of the energy norm (12), we get the desired result.

Case 2. $\kappa p\mu < \frac{1}{2}$ i.e. $p < \frac{1}{2\kappa\mu}$ (pre-asymptotic case), $\Delta = \{0, \kappa p\varepsilon, \kappa p\mu, 1-\kappa p\mu, 1-\kappa p\varepsilon, 1\}$

The mesh consists of five elements I_i, $i = 1, 2, \ldots, 5$ and we decompose $\overrightarrow{u}$ as in (18):

$$\overrightarrow{u} = \overrightarrow{w} + A^-\overrightarrow{u}^- + A^+\overrightarrow{u}^+ + \overrightarrow{r}.$$

The expansion order M is chosen as the integer part of $\eta\kappa p/4$ (and for notational convenience we will simply write $4M = \eta\kappa p$) where $\eta > 0$ is a fixed parameter satisfying

$$\frac{1}{2}\eta\overline{K}_1 < 1, \qquad \frac{1}{2}\eta K_1 =: \delta < \frac{1}{2}$$

with $\overline{K}_1$ and K_1 the constants from Theorems 4 and 7, respectively. The choice of η guarantees that as $\kappa p\mu < \frac{1}{2}$, we have

$$4M\varepsilon\overline{K}_1 < 4M\mu\overline{K}_1 \leq \eta\kappa p\mu\overline{K}_1 < \frac{1}{2}\eta\overline{K}_1 < 1$$

and

$$4M\varepsilon K_1 < 4M\mu K_1 \leq \eta\kappa p\mu K_1 < \frac{1}{2}\eta K_1 =: \delta < \frac{1}{2}.$$

Thus the assumptions of Theorem 4 are satisfied and the remainder $\overrightarrow{r}$ is small by Theorem 7 – in particular, we have

$$\|\overrightarrow{r}\|_{E,\Omega} \leq C\mu^2(4M\mu K_1)^{2M} \leq C\mu^2\delta^{2M} \leq C\mu^2\delta^{\eta\kappa p/2} \leq C\mu^2e^{-\beta_2 p}, \tag{44}$$

where $\beta_2 = |\ln q_2|$, $q_2 = \delta^{\eta\kappa/2} < 1$.

We next analyze the approximation of each of the remaining three terms in the decomposition (18).

For the approximation of $\overrightarrow{w}$, we have, by Theorem 8, that there exists $\mathcal{I}_p\overrightarrow{w} \in \overrightarrow{S}(\kappa,p)$ such that $\overrightarrow{w} = \mathcal{I}_p\overrightarrow{w}$ on $\partial\Omega$ and for $0 \le s \le p$

$$\left\|(\overrightarrow{w} - \mathcal{I}_p\overrightarrow{w})'\right\|_{0,\Omega}^2 \le \frac{(p-s)!}{(p+s)!}\left\|\overrightarrow{w}^{(s+1)}\right\|_{0,\Omega}^2 \le \frac{(p-s)!}{(p+s)!} C K^{2(s+1)}\left((s+1)!\right)^2,$$

where we used Theorem 4. Choosing $s = \overline{\lambda}p$, for some $\overline{\lambda} \in (0,1]$, and using Lemma 10, we get

$$\begin{aligned}
\left\|(\overrightarrow{w} - \mathcal{I}_p\overrightarrow{w})'\right\|_{0,\Omega}^2 &\le \left[\frac{\left(1-\overline{\lambda}\right)^{(1-\overline{\lambda})}}{\left(1+\overline{\lambda}\right)^{(1+\overline{\lambda})}}\right]^p p^{-2\overline{\lambda}p} e^{2\overline{\lambda}p+1} C K_2^{2\overline{\lambda}p+2}\left[\left(\overline{\lambda}p+1\right)^{\overline{\lambda}p+1+1/2} e^{-\overline{\lambda}p-1}\right]^2 \\
&\le C\left(\overline{\lambda}p+1\right)^3\left[\frac{\left(1-\overline{\lambda}\right)^{(1-\overline{\lambda})}}{\left(1+\overline{\lambda}\right)^{(1+\overline{\lambda})}}\right]^p K_2^{2\overline{\lambda}p}\left(\frac{1+\overline{\lambda}p}{p}\right)^{2\lambda p} \\
&\le C\left(\overline{\lambda}p+1\right)^3\left[\frac{\left(1-\overline{\lambda}\right)^{(1-\overline{\lambda})}}{\left(1+\overline{\lambda}\right)^{(1+\overline{\lambda})}}\right]^p K_2^{2\overline{\lambda}p}\overline{\lambda}^{2\overline{\lambda}p}\left[\left(1+\frac{1}{\overline{\lambda}p}\right)^{\overline{\lambda}p}\right]^2 \\
&\le Cp^3\left[\frac{\left(1-\overline{\lambda}\right)^{(1-\overline{\lambda})}}{\left(1+\overline{\lambda}\right)^{(1+\overline{\lambda})}}\left(K_2\overline{\lambda}\right)^{2\overline{\lambda}}\right]^p.
\end{aligned}$$

Thus, we choose $\overline{\lambda} = 1/K_2 \in (0,1)$ and we have

$$\left\|(\overrightarrow{w} - \mathcal{I}_p\overrightarrow{w})'\right\|_{0,\Omega}^2 \le Cp^3 e^{-\beta_3 p}, \tag{45}$$

where $\beta_3 = |\ln q_3|$, $q_3 = \frac{(1-\overline{\lambda})^{(1-\overline{\lambda})}}{(1+\overline{\lambda})^{(1+\overline{\lambda})}} < 1$. Repeating the previous argument for the L^2 norm of $(\overrightarrow{w} - \mathcal{I}_p\overrightarrow{w})$, we get, using (41),

$$\left\|\overrightarrow{w} - \mathcal{I}_p\overrightarrow{w}\right\|_{0,\Omega}^2 \le Cpe^{-\beta_3 p}, \tag{46}$$

with $C > 0$ independent of ε and μ.

We now approximate the boundary layers. We will only consider $A^-\overrightarrow{u}^-$, since $A^+\overrightarrow{u}^+$ is completely analogous, and we will concentrate on the approximation of $\overrightarrow{u}^-$, since $A^- = \text{diag}(-w_1(0), -w_2(0))$ is bounded independently of ε and μ (see Remark 1). Recall that

$$\overrightarrow{u}^- = [u_1^-, u_2^-]^T = [u_{1,\varepsilon}^- + u_{1,\mu}^-, u_{2,\varepsilon}^- + u_{2,\mu}^-]^T,$$

hence, we will construct separate approximations for $u_{1,\varepsilon}^-$ on the intervals $\overline{I}_1 = I_1 = [0, \kappa p\varepsilon]$, $\overline{I}_2 = \cup_{i=2}^5 I_i$ and for $u_{1,\mu}^-$ on the intervals $\overline{I}_3 = I_1 \cup I_2 = [0, \kappa p\mu]$, $\overline{I}_4 = \cup_{i=3}^5 I_i$. The same will be done for u_2^-.

By Theorem 8 there exists $\mathcal{I}_p u_{1,\varepsilon}^- \in S(\kappa,p)$ such that $\mathcal{I}_p u_{1,\varepsilon}^- = u_{1,\varepsilon}^-$ on $\partial\overline{I}_1$ and for $0 \le s \le p$

$$\left\|\left(u_{1,\varepsilon}^- - \mathcal{I}_p u_{1,\varepsilon}^-\right)'\right\|_{0,\overline{I}_1}^2 \le (\kappa p\varepsilon)^{2s}\frac{(p-s)!}{(p+s)!}\left\|\left(u_{1,\varepsilon}^-\right)^{(s+1)}\right\|_{0,\overline{I}_1}^2. \tag{47}$$

By Lemma 6,

$$\begin{aligned}\left\|\left(u_{1,\varepsilon}^{-}\right)^{(s+1)}\right\|_{0,\overline{I}_1}^2 &= \int_0^{\kappa p\varepsilon}\left|\left(u_{1,\varepsilon}^{-}\right)^{(s+1)}(x)\right|^2 dx \\ &\leq C\kappa p\varepsilon K^{2(s+1)}\max\{s+1,\varepsilon^{-1}\}^{2(s+1)}\max_{x\in[0,\kappa p\varepsilon]}\{e^{-x\alpha/\varepsilon}\}, \qquad (48)\end{aligned}$$

for $s+1=0,1,...,q<\alpha/\mu$. Since $\kappa p\varepsilon<\kappa p\mu<1/2$, i.e. $s\leq p<\frac{1}{2\kappa\mu}<\frac{1}{2\kappa\varepsilon}$, we have $\max\{s+1,\varepsilon^{-1}\}^{2(s+1)}=\varepsilon^{-2(s+1)}$ and (47), (48) give

$$\begin{aligned}\left\|\left(u_{1,\varepsilon}^{-}-\mathcal{I}_p u_{1,\varepsilon}^{-}\right)'\right\|_{0,\overline{I}_1}^2 &\leq (\kappa p\varepsilon)^{2s}\frac{(p-s)!}{(p+s)!}C\kappa p\varepsilon K^{2(s+1)}\varepsilon^{-2(s+1)} \\ &\leq CK^{2(s+1)}\kappa^{2s+1}p^{2s+1}\varepsilon^{-1}\frac{(p-s)!}{(p+s)!}.\end{aligned}$$

Choosing $s=\widetilde{\lambda}p$ for some $\widetilde{\lambda}\in(0,1)$, and using Lemma 10, we further obtain

$$\begin{aligned}\left\|\left(u_{1,\varepsilon}^{-}-\mathcal{I}_p u_{1,\varepsilon}^{-}\right)'\right\|_{0,\overline{I}_1}^2 &\leq CK^{2(\widetilde{\lambda}p+1)}\kappa^{2\widetilde{\lambda}p+1}p^{2\widetilde{\lambda}p+1}\varepsilon^{-1}\frac{(p-\widetilde{\lambda}p)!}{(p+\widetilde{\lambda}p)!} \\ &\leq CK^{2(\widetilde{\lambda}p+1)}\kappa^{2\widetilde{\lambda}p+1}p^{2\widetilde{\lambda}p+1}\varepsilon^{-1}\left[\frac{\left(1-\widetilde{\lambda}\right)^{(1-\widetilde{\lambda})}}{\left(1+\widetilde{\lambda}\right)^{(1+\widetilde{\lambda})}}\right]^p p^{-2\widetilde{\lambda}p}e^{2\widetilde{\lambda}p+1} \\ &\leq CeK^2\kappa p\varepsilon^{-1}\left[\frac{\left(1-\widetilde{\lambda}\right)^{(1-\widetilde{\lambda})}}{\left(1+\widetilde{\lambda}\right)^{(1+\widetilde{\lambda})}}\right]^p (Ke\kappa)^{2\widetilde{\lambda}p} \\ &\leq Cp\varepsilon^{-1}e^{-\beta_4 p}, \qquad (49)\end{aligned}$$

where $\beta_4=|\ln q_4|$, $q_4=\frac{(1-\widetilde{\lambda})^{(1-\widetilde{\lambda})}}{(1+\widetilde{\lambda})^{(1+\widetilde{\lambda})}}<1$, *provided* we choose $\kappa=\frac{1}{eK}$. Now, on the interval $\overline{I}_2=[\kappa p\varepsilon,1]$, $u_{1,\varepsilon}^{-}$ is already exponentially small, and by Lemma 6

$$\left\|\left(u_{1,\varepsilon}^{-}\right)'\right\|_{0,\overline{I}_2}^2=\int_{\kappa p\varepsilon}^1\left|\left(u_{1,\varepsilon}^{-}\right)'\right|^2dx\leq C\varepsilon^{-2}\left(1-\kappa p\varepsilon\right)\max_{x\in\overline{I}_2}\left\{e^{-2x\alpha/\varepsilon}\right\}\leq C\varepsilon^{-2}e^{-2\kappa p\alpha},$$

with $C>0$ independent of ε and μ. Thus, we approximate $u_{1,\varepsilon}^{-}$ by its linear interpolant $\mathcal{I}_1u_{1,\varepsilon}^{-}$, and we have

$$\left\|\left(u_{1,\varepsilon}^{-}-\mathcal{I}_1u_{1,\varepsilon}^{-}\right)'\right\|_{0,\overline{I}_2}^2\leq\left\|\left(u_{1,\varepsilon}^{-}\right)'\right\|_{0,\overline{I}_2}^2+\left\|\left(\mathcal{I}_1u_{1,\varepsilon}^{-}\right)'\right\|_{0,\overline{I}_2}^2\leq C\varepsilon^{-2}e^{-2\kappa p\alpha},$$

which along with (49) give

$$\left\|\left(u_{1,\varepsilon}^{-}-\mathcal{I}_pu_{1,\varepsilon}^{-}\right)'\right\|_{0,\Omega}^2\leq Cp\varepsilon^{-2}e^{-\beta_5 p}, \qquad (50)$$

for some $\beta_5 > 0$, independent of ε and μ. Repeating the previous arguments for the L^2 norm of $\left(u_{1,\varepsilon}^{-} - \mathcal{I}_p u_{1,\varepsilon}^{-}\right)$, we get

$$\left\| u_{1,\varepsilon}^{-} - \mathcal{I}_p u_{1,\varepsilon}^{-} \right\|_{0,\Omega}^2 \le C e^{-\beta_6 p}, \tag{51}$$

for some $\beta_6 > 0$, independent of ε and μ.

The approximation of $u_{1,\mu}^{-}$ on the intervals $\overline{I}_3 = I_1 \cup I_2 = [0, \kappa p \mu]$, $\overline{I}_4 = \cup_{i=3}^{5} I_i$ is completely analogous (and the details are ommitted), yielding

$$\left\| \left(u_{1,\mu}^{-} - \mathcal{I}_p u_{1,\mu}^{-}\right)' \right\|_{0,\Omega}^2 \le C p \mu^{-2} e^{-\beta_7 p} \tag{52}$$

and

$$\left\| u_{1,\mu}^{-} - \mathcal{I}_p u_{1,\mu}^{-} \right\|_{0,\Omega}^2 \le C e^{-\beta_8 p}, \tag{53}$$

for some $\beta_7, \beta_8 > 0$, independent of ε and μ.

We now turn our attention to the approximation of $u_2^{-} = u_{2,\varepsilon}^{-} + u_{2,\mu}^{-}$, which is achieved in a similar fashion as above. On the interval $\overline{I}_1 = I_1 = [0, \kappa p \varepsilon]$, we have by Theorem 8

$$\left\| \left(u_{2,\varepsilon}^{-} - \mathcal{I}_p u_{2,\varepsilon}^{-}\right)' \right\|_{0,\overline{I}_1}^2 \le (\kappa p \varepsilon)^{2s} \frac{(p-s)!}{(p+s)!} \left\| \left(u_{2,\varepsilon}^{-}\right)^{(s+1)} \right\|_{0,\overline{I}_1}^2, \quad 0 \le s \le p$$

and by Lemma 6,

$$\begin{aligned} \left\| \left(u_{2,\varepsilon}^{-}\right)^{(s+1)} \right\|_{0,\overline{I}_1}^2 &= \int_0^{\kappa p \varepsilon} \left| \left(u_{2,\varepsilon}^{-}\right)^{(s+1)} (x) \right|^2 dx \\ &\le C \kappa p \varepsilon K^{2(s+1)} \max\{(s+1)^{2(s+1)}, \mu^{-4} \varepsilon^{-2(s+1)+4}\} \max_{x \in [0, \kappa p \varepsilon]} \{e^{-x\alpha/\varepsilon}\} \\ &\le C \kappa p \varepsilon K^{2(s+1)} \max\{(s+1)^{2(s+1)}, \mu^{-2} \varepsilon^{-2s}\}. \end{aligned}$$

Since $\kappa p \varepsilon < \kappa p \mu < 1/2$, we have $\max\{s+1, \varepsilon^{-1}\}^{2(s+1)} = \varepsilon^{-2(s+1)}$ and hence

$$\left\| \left(u_{2,\varepsilon}^{-} - \mathcal{I}_p u_{2,\varepsilon}^{-}\right)' \right\|_{0,\overline{I}_1}^2 \le (\kappa p \varepsilon)^{2s} \frac{(p-s)!}{(p+s)!} C \kappa p \varepsilon K^{2(s+1)} \mu^{-2} \varepsilon^{-2s} \le C \mu^{-2} (\kappa p K)^{2s+1} \frac{(p-s)!}{(p+s)!}.$$

Choosing $s = \widehat{\lambda} p$ for some $\widehat{\lambda} \in (0,1)$, and arguing as in the derivation of (49) above, we can show that

$$\left\| \left(u_{2,\varepsilon}^{-} - \mathcal{I}_p u_{2,\varepsilon}^{-}\right)' \right\|_{0,\overline{I}_1}^2 \le C p \mu^{-2} e^{-\beta_9 p}, \tag{54}$$

for some $\beta_9 > 0$, independent of ε and μ, *provided* we choose $\kappa = \frac{1}{eK}$. On the interval $\overline{I}_2 = [\kappa p \varepsilon, 1]$, $u_{1,\varepsilon}^{-}$ is already exponentially small, and by Lemma 6

$$\left\| \left(u_{2,\varepsilon}^{-}\right)' \right\|_{0,\overline{I}_2}^2 = \int_{\kappa p \varepsilon}^{1} \left| \left(u_{2,\varepsilon}^{-}\right)' \right|^2 dx \le C \mu^{-2} (1 - \kappa p \varepsilon) \max_{x \in \overline{I}_2} \left\{ e^{-2x\alpha/\varepsilon} \right\} \le C \mu^{-2} e^{-2\kappa p \alpha}.$$

Therefore, we approximate $u^-_{2,\varepsilon}$ by its linear interpolant $\mathcal{I}_1 u^-_{2,\varepsilon}$, and we have

$$\left\| \left(u^-_{2,\varepsilon} - \mathcal{I}_1 u^-_{2,\varepsilon} \right)' \right\|^2_{0,\overline{I}_2} \leq \left\| \left(u^-_{2,\varepsilon} \right)' \right\|^2_{0,\overline{I}_2} + \left\| \left(\mathcal{I}_1 u^-_{1,\varepsilon} \right)' \right\|^2_{0,\overline{I}_2} \leq C\mu^{-2} e^{-2\kappa p \alpha},$$

which along with (54) give

$$\left\| \left(u^-_{2,\varepsilon} - \mathcal{I}_p u^-_{2,\varepsilon} \right)' \right\|^2_{0,\Omega} \leq Cp\mu^{-2} e^{-\beta_{10} p}, \tag{55}$$

for some $\beta_{10} > 0$, independent of ε and μ. In a similar fashion, we obtain

$$\left\| u^-_{2,\varepsilon} - \mathcal{I}_p u^-_{2,\varepsilon} \right\|^2_{0,\Omega} \leq Ce^{-\beta_{11} p}, \tag{56}$$

for some $\beta_{11} > 0$, independent of ε and μ. The same arguments work for the approximation of $u^-_{2,\mu}$, allowing us to get estimates analogous to (55) and (56). Thus, combining (50), (51), (52), (53), (55), (56) and the analogous estimates for $u^-_{2,\mu}$, we have

$$\left\| \left(u^-_1 - \mathcal{I}_p u^-_1 \right)' \right\|^2_{0,\Omega} \leq Cp\varepsilon^{-2} e^{-\beta p} \ , \ \left\| u^-_1 - \mathcal{I}_p u^-_1 \right\|^2_{0,\Omega} \leq Ce^{-\beta p}, \tag{57}$$

and

$$\left\| \left(u^-_2 - \mathcal{I}_p u^-_2 \right)' \right\|^2_{0,\Omega} \leq Cp\mu^{-2} e^{-\beta p} \ , \ \left\| u^-_2 - \mathcal{I}_p u^-_2 \right\|^2_{0,\Omega} \leq Ce^{-\beta p}, \tag{58}$$

for some $\beta > 0$, independent of ε and μ. Using the same techniques, similar bounds can be obtained for $\overrightarrow{u}^+$.

Combining (44), (46), (57), (58) and the analogous bounds for $\overrightarrow{u}^+$, we have

$$\begin{aligned}
\|\overrightarrow{u} - \mathcal{I}_p \overrightarrow{u}\|^2_{0,\Omega} &= \left\| \left(\overrightarrow{w} + A^- \overrightarrow{u}^- + A^+ \overrightarrow{u}^+ + \overrightarrow{r} \right) - \left(\mathcal{I}_p \overrightarrow{w} + A^- \mathcal{I}_p \overrightarrow{u}^- + A^+ \mathcal{I}_p \overrightarrow{u}^+ + \overrightarrow{r} \right) \right\|^2_{0,\Omega} \\
&\leq \|\overrightarrow{w} - \mathcal{I}_p \overrightarrow{w}\|^2_{0,\Omega} + C \left\{ \|\overrightarrow{u}^- - \mathcal{I}_p \overrightarrow{u}^-\|^2_{0,\Omega} + \left\| \overrightarrow{u}^+ - \mathcal{I}_p \overrightarrow{u}^+ \right\|^2_{0,\Omega} \right\} + \|\overrightarrow{r}\|^2_{0,\Omega} \\
&\leq Cpe^{-\beta p},
\end{aligned}$$

for some $\beta > 0$, independent of ε and μ. Similarly,

$$\begin{aligned}
|u_1 - \mathcal{I}_p u_1|^2_{1,\Omega} &\leq |w_1 - \mathcal{I}_p w_1|^2_{1,\Omega} + C \left\{ \left| u^-_{1,\varepsilon} - \mathcal{I}_p u^-_{1,\varepsilon} \right|^2_{1,\Omega} + \left| u^-_{1,\mu} - \mathcal{I}_p u^-_{1,\mu} \right|^2_{1,\Omega} \right\} + \\
&\quad + C \left\{ \left| u^+_{1,\varepsilon} - \mathcal{I}_p u^+_{1,\varepsilon} \right|^2_{1,\Omega} + \left| u^+_{1,\mu} - \mathcal{I}_p u^+_{1,\mu} \right|^2_{1,\Omega} \right\} + |r|^2_{1,\Omega} \\
&\leq C\varepsilon^{-2} p^3 e^{-\beta p},
\end{aligned}$$

and

$$|u_2 - \mathcal{I}_p u_2|^2_{1,\Omega} \leq C\mu^{-2} p^3 e^{-\beta p},$$

so that

$$\begin{aligned}
\|\overrightarrow{u} - \mathcal{I}_p \overrightarrow{u}\|^2_{E,\Omega} &= \varepsilon^2 \left| u_1 - \mathcal{I}_p u_1 \right|^2_{1,\Omega} + \mu^2 \left| u_2 - \mathcal{I}_p u_2 \right|^2_{1,\Omega} + \alpha \left(\|u_1 - \mathcal{I}_p u_1\|^2_{0,\Omega} + \|u_2 - \mathcal{I}_p u_2\|^2_{0,\Omega} \right) \\
&\leq Cp^3 e^{-\beta p}
\end{aligned}$$

as desired.

Case 3. $\kappa p\varepsilon < \frac{1}{2}$ and $\kappa p\mu \geq \frac{1}{2}$, i.e. $\frac{1}{2\kappa\mu} \leq p < \frac{1}{2\kappa\varepsilon}$ ('semi'-asymptotic case), $\Delta = \{0, \kappa p\varepsilon, 1-\kappa p\varepsilon, 1\}$.

In this case p is in the range where the boundary layers involving μ are resolved without any mesh refinement, but those involving ε still require mesh refinement.

As we are still assuming that $4M\mu$ is small (see Remark 2), the assumptions of Theorems 4 and 7 hold, and we will omit the discussion pertaining to the approximation of the smooth part and the remainder (since it is similar to Case 2). For the boundary layers we will only consider $u_2^- = u_{2,\varepsilon}^- + u_{2,\mu}^-$ since the approximation of u_1^- and $\overrightarrow{u}^+ = [u_1^+, u_2^+]^T$ is completely analogous. For $u_{2,\varepsilon}^-$ we will construct separate approximations on $\overline{I}_1 = [0, \kappa p\varepsilon]$ and $\overline{I}_2 = [\kappa p\varepsilon, 1]$, as in Case 2, while $u_{2,\mu}^-$ will be approximated on $\overline{\Omega} = [0,1]$. For the latter, we have by Lemma 6,

$$\left\| \left(u_{2,\mu}^-\right)^{(n)} \right\|_{0,\Omega}^2 \leq CK^{2n} \max\{n, \mu^{-1}\}^{2n} \ , \ n = 0, 1, ..., q < \alpha/\mu,$$

while by Theorem 8 there exists $\mathcal{I}_p u_{2,\mu}^- \in S(\kappa, p)$ such that $\mathcal{I}_p u_{2,\mu}^- = u_{2,\mu}^-$ on $\partial\Omega$ and for $s = 0, 1, ..., p$

$$\begin{aligned} \left\| \left(\mathcal{I}_p u_{2,\mu}^- - u_{2,\mu}^-\right)' \right\|_{0,\Omega}^2 &\leq \frac{(p-s)!}{(p+s)!} \left\| \left(u_{2,\mu}^-\right)^{(s+1)} \right\|_{0,\Omega}^2 \\ &\leq \frac{(p-s)!}{(p+s)!} CK^{2n} \max\{s+1, \mu^{-1}\}^{2(s+1)}. \end{aligned}$$

Choose $s = \widehat{\lambda} p$, for some $\widehat{\lambda} \in (0,1]$. Then, since $p \geq 1/(2\kappa\mu)$ we have

$$\max\{s+1, \mu^{-1}\}^{2(s+1)} = \max\{\widehat{\lambda} p + 1, \mu^{-1}\}^{2(\widehat{\lambda} p+1)} = \left(\widehat{\lambda} p + 1\right)^{2(\widehat{\lambda} p+1)}$$

and

$$\left\| \left(\mathcal{I}_p u_{2,\mu}^- - u_{2,\mu}^-\right)' \right\|_{0,\Omega}^2 \leq \frac{\left(p - \widehat{\lambda} p\right)!}{\left(p + \widehat{\lambda} p\right)!} CK^{2(\widehat{\lambda} p+1)} \left(\widehat{\lambda} p + 1\right)^{2(\widehat{\lambda} p+1)}$$

from which, following the same steps as in Case 1, we obtain

$$\left\| \left(\mathcal{I}_p u_{2,\mu}^- - u_{2,\mu}^-\right)' \right\|_{0,\Omega}^2 \leq Cp^2 e^{-|\ln q| p}, \tag{59}$$

with $q = \frac{\left(1-\widehat{\lambda}\right)^{1-\widehat{\lambda}}}{\left(1+\widehat{\lambda}\right)^{1+\widehat{\lambda}}} < 1$, provided we choose $\widehat{\lambda} = 1/(eK) < 1$. A similar argument yields

$$\left\| \mathcal{I}_p u_{2,\mu}^- - u_{2,\mu}^- \right\|_{0,\Omega}^2 \leq Ce^{-|\ln q| p}. \tag{60}$$

The approximation of $u_{2,\varepsilon}^-$ on $\overline{I}_1 = [0, \kappa p\varepsilon]$ and $\overline{I}_2 = [\kappa p\varepsilon, 1]$ is identical to Case 2 (and the details are omitted), resulting in bounds analogous to (55) and (56), which in turn give us the estimates (58). The desired result is, thus, obtained by arguing as in Case 2 above. □

Using the above theorem and the quasioptimality result (16) we have the following.

Corollary 12. *Let* $\vec{u}$ *be the solution to* (1),(3) *and let* $\vec{u}_{FE} \in \vec{S}_0^p(\Delta)$ *be the solution to* (15). *Then exist constants* $\kappa, C, \sigma > 0$ *depending* only *on the input data* $\vec{f}$ *and* A *such that*

$$\|\vec{u} - \vec{u}_{FE}\|_{E,\Omega} \leq Cp^{3/2}e^{-\sigma p}.$$

The above result shows that as $p \to \infty$ the method converges at an exponential rate, independently of the singular perturbation parameters ε and μ, when the error is measured in the energy norm. In the numerical study from [20] it was observed that the method not only converges at the above rate, but as $\varepsilon, \mu \to 0$ the performance improves. In particular, the following estimate was observed:

$$\|\vec{u} - \vec{u}_{FE}\|_{E,\Omega} \leq C \max\{\varepsilon, \mu\}^{1/2} e^{-\sigma p} = C\mu^{1/2}e^{-\sigma p}. \tag{61}$$

This was the case for the scalar problem with constant coefficients and polynomial right hand side studied in [16]. It is interesting to note that in [20] the above rate was observed for the variable coefficient problem as well, even though for this case no exact solution was available and the errors were computed using a reference solution.

4 Concluding Remarks

We have studied the approximation of a coupled system of singularly perturbed reaction-diffusion equations, by the finite element method, focusing on the case when the singular perturbation parameters ε and μ satisfy $0 < \varepsilon < \mu << 1$. We showed that under the assumption of analytic input data, the *hp* version on the variable five element mesh $\{0, \kappa p\varepsilon, \kappa p\mu, 1-\kappa p\mu, 1-\kappa p\varepsilon, 1\}$ yields exponential convergence, independently of ε and μ, as $p \to \infty$, when the error is measured in the energy norm. The constant κ in the mesh was shown to depend on the constant of analyticity of the input data.

The cases when $0 < \varepsilon = \mu < 1$ and $0 < \varepsilon < \mu = 1$ are presented in [21] and [22], respectively.

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