SYMMETRIC EIGENFACES

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Abstract. Over the years, mathematicians and computer scientists have produced an extensive body of work in the area of facial analysis. Several facial analysis algorithms have been based on mathematical concepts such as the singular value decomposition (SVD). The SVD is generalized in this paper to take advantage of the mirror symmetry that is inherent in faces, thereby developing a new facial recognition algorithm: the symmetry preserving singular value decomposition (SPSVD). The SPSVD recognizes faces using half the total computations employed by the conventional SVD. Moreover, the SPSVD provides more accurate recognition, even in the presence of noise in the form of light variation and/or facial occlusion.

Key words. Facial analysis, eigenfaces, reflection, symmetry, SVD, PCA, SPSVD

AMS subject classifications. 15A18, 65F15

1. Introduction. Facial recognition is a technology that is fast becoming critical for contemporary security applications. For instance, many airports employ facial recognition algorithms in their closed-circuit televisions to identify - in real-time - potential terrorists. Future applications of facial recognition may include the use of these algorithms in facilitating user identification by, e.g., automated teller machines (ATMs).

Many facial recognition algorithms are based on the singular value decomposition (SVD) [3,4,7]. The SVD takes an image of a given face and compares it to facial images stored in a database, using a reduced representation of the face. The representation in the database that most closely matches the face of the person being scanned is returned, as is a measure of difference between the two faces. Thus, this approach can provide an estimate of the probability that a match has been detected. To increase efficiency, the SVD may also employ a procedure of projecting a face onto a low rank representation of the database.

Misidentification may occur with either of these methods especially in the case when there are lighting discrepancies or in the case when parts of the face are occluded (hidden). For example, if a person approaches an ATM where the sun is shining on only one-half of his/her face, then the computer may misidentify the person. In order to deal with the problems of light variation and/or facial occlusion associated with the SVD, a new algorithm the symmetry-preserving singular value decomposition (SPSVD) is constructed. This algorithm takes advantage of the inherent mirror symmetry of a face to "average-out" discrepancies due to light variation and/or facial occlusion.

In this paper, we assume that faces are reflective symmetric along the midline of the face. Mathematically, this assumption implies that the left face is approximately equal to the right face (in reverse order). A plethora of work has been done to correctly find and orient faces [5, 8, 10].

Once the faces are correctly oriented, the SPSVD can be utilized to correctly identify faces from a training set of images. This algorithm is based on the works by Kirby and Sirovich [3, 4], as well as Turk and Pentland [7]. In this research, principal component analysis (PCA) is employed as a facial recognition algorithm. The left eigenvectors (eigenfaces) are used as a basis for the face space of the system. This

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research is extended here to take advantage of the inherent symmetry of faces to create a "symmetric" SVD that forms a basis for the symmetric face space that requires half the storage of the original face space. It should be noted that PCA is mathematically equivalent to the mean-adjusted SVD algorithm used in this paper.

Symmetry has been used in a series of facial recognition algorithms to increase the accuracy and efficiency of the underlying algorithm. For instance, Yang and Ding [9] split the facial data set into an even and odd set. Then they apply PCA on each set to form two bases to extract information to be used in facial recognition. The SPSVD is similar in that it uses the inherent symmetry of the faces to build an orthogonal basis for the data set. However, here it is proven that the SPSVD provides a basis for the best symmetric approximation to the face space.

This paper is organized as follows. Section 2 defines the SVD and SPSVD. Section 3 outlines the algorithm for facial recognition using the SVD and SPSVD. Section 4 describes the computational results of applying the SVD and SPSVD to a series of faces with increasing facial occlusions and light variations. Finally, Section 5 gives concluding remarks.

2. Methods. An $m \times n$ dimensional photograph can be thought of as an $m \times n$ dimensional matrix of intensity values where the (i, j)-th element represents the intensity value of the (i, j)-th position or pixel value of the photograph. An alternate representation of a given photograph is to construct an mn-dimensional vector created by stacking each column of the photograph on top of each other. Therefore, any $(m \times n)$ photograph can be represented as a point in an mn-dimensional space known as the "face space". In general, faces of the same person will not be spread out throughout the face space, but instead they will be clustered in certain areas [7]. Therefore, a training set of k photographs in the original mn-dimensional space, where $k << mn$ may accurately be represented by a lower dimensional subspace of the original face space. The goal of both the SVD and SPSVD is to take advantage of this low-dimensionality by projecting a new photograph onto this low-dimensional subspace and finding the cluster that it is closest to; thereby identifying the face from the original training set.

2.1. SVD. For a given training set of k images, each represented as an mn-dimensional vectors P_i, consider the $(mn \times k)$ matrix

$$\mathbf{P} = (P_1 - \Psi, P_2 - \Psi, \ldots, P_k - \Psi)$$

where $\Psi = \frac{1}{k} \sum_{i=1}^{k} P_i$ is the average face (derived from the original training set). Then the SVD may be applied to

$$\mathbf{P} = \mathbf{USV}^T$$

where $\mathbf{U}$ and $\mathbf{V}$ are orthogonal matrices and $\mathbf{S} = \text{diag}(s_1, s_2, \ldots, s_n)$ is a diagonal matrix with non-negative terms in decreasing order. These left singular vectors, or "eigenfaces", u_i of $\mathbf{U}$ form an orthogonal basis for the face space ranked by the corresponding singular value s_i. Therefore, the best ℓ-dimensional representation of the face space may be formed by the first ℓ left singular vectors ranked by the first ℓ singular values.

2.2. SPSVD. The SPSVD takes advantage of the symmetry that is inherent in faces. By assuming that each face is centered along the midline of the face, the left half of the face should be approximately equal to the mirror image of the right half

of the face. Mathematically speaking, this notion implies that if a face is represented by an $m \times n$ matrix of pixel values, where n is an even number, then the first $n/2$ columns of the matrix should be approximately equal to the last $n/2$ columns of the matrix in reverse order. This idea implies that faces are inherently redundant. The SPSVD takes advantage of this redundancy by producing a "symmetric face space" that is half the dimension of the conventional face space.

The symmetric face space can be calculated in the following way [6]. For a given training set of k images, each represented as an mn-dimensional vectors P_i (where n is assumed to be even), consider the $(\frac{mn}{2} \times k)$ matrix

$$\mathbf{P} = (\hat{P}_1 - \Psi, \hat{P}_2 - \Psi, \ldots, \hat{P}_k - \Psi)$$

where the symmetric face

$$\hat{P}_i = \frac{1}{2}\left[P_i\left(1:\frac{mn}{2}\right) + \mathbf{R}P_i\left(\frac{mn}{2}+1:mn\right)\right]$$

where

$$\mathbf{R} = \begin{pmatrix} 0 & & \mathbf{I}_m \\ & \cdot^{\cdot^{\cdot}} & \\ \mathbf{I}_m & & 0 \end{pmatrix},$$

$\mathbf{I}_m$ is the m-dimensional identity matrix, and $\Psi = \frac{1}{k}\sum_{i=1}^{k}\hat{P}_i$ is the average symmetric face. Then the SVD may be applied to

$$\mathbf{P} = \frac{1}{2}(\mathbf{P}_L + \mathbf{R}\mathbf{P}_R) = \mathbf{U}\mathbf{S}\mathbf{V}^T$$

to form the SPSVD where $\mathbf{U}$ and $\mathbf{V}$ are orthogonal matrices and $\mathbf{S} = \text{diag}(s_1, s_2, \ldots, s_n)$ is a diagonal matrix with non-negative terms in decreasing order. In addition, $\mathbf{P}_L$ is the first $\frac{mn}{2}$ rows of $\mathbf{P}$ and $\mathbf{P}_R$ is the last $\frac{mn}{2}$ rows of $\mathbf{P}$. Notice, that the left symmetric singular vectors, or "symmetric eigenfaces" u_i of $\mathbf{U}$ are half the size of the conventional left singular vectors. In addition, these symmetric eigenfaces form an orthogonal basis for the symmetric face space ranked by the corresponding singular value s_i. Moreover, the best ℓ-dimensional symmetric representation of the symmetric face space may be formed by the first ℓ left symmetric singular vectors ranked by the first ℓ symmetric singular values. This is proved with the following theorem.

THEOREM 2.1.
For a correctly aligned set of training images $\mathbf{P}$, *where*

$$\mathbf{R}\mathbf{P}_L = \mathbf{P}_R + \mathbf{E},$$

the best symmetric approximation

$$\hat{\mathbf{P}} = \begin{pmatrix} \hat{\mathbf{P}}_L \\ \hat{\mathbf{P}}_R \end{pmatrix}$$

to the face space with regards to both the 2-norm and Frobenius norm can be found by minimizing

$$\min_{\hat{\mathbf{P}}_R = \mathbf{R}\hat{\mathbf{P}}_L} \left\| \begin{pmatrix} \mathbf{P}_L \\ \mathbf{P}_R \end{pmatrix} - \begin{pmatrix} \hat{\mathbf{P}}_L \\ \hat{\mathbf{P}}_R \end{pmatrix} \right\|^2.$$

The solution to this minimization problem can be calculated with the SVD of

$$\mathbf{U}\mathbf{S}\mathbf{V}^T = \left(\begin{array}{c} \hat{\mathbf{P}}_L \\ \hat{\mathbf{P}}_R \end{array} \right)$$

where

$$\mathbf{U} = \frac{1}{\sqrt{2}} \left(\begin{array}{c} \mathbf{U}_L \\ \mathbf{U}_R \end{array} \right), \quad \mathbf{S} = \sqrt{2}\mathbf{S}_L, \quad \mathbf{V} = \mathbf{V}_L,$$

and

$$\mathbf{U}_R = \mathbf{R}\mathbf{U}_L,$$

with

$$\mathbf{U}_L \mathbf{S}_L \mathbf{V}_L^T = \frac{1}{2}(\mathbf{P}_L + \mathbf{R}\mathbf{P}_R).$$

Moreover, the best rank-ℓ symmetric approximation to the face space is

$$\sum_{j=1}^{\ell} s_j \mathbf{u}_j \mathbf{v}_j^T$$

where $\mathbf{u}_j$ and $\mathbf{v}_j$ are the jth column of $\mathbf{U}$ and $\mathbf{V}$, respectively, and s_j is the jth singular value of $\mathbf{S}$ ordered decreasingly.

    Proof. Consider the orthogonal matrix

$$\mathbf{B} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{R} & -\mathbf{I} \\ \mathbf{I} & \mathbf{R} \end{pmatrix}.$$

Then

$$\left\| \left(\begin{array}{c} \mathbf{P}_L \\ \mathbf{P}_R \end{array} \right) - \left(\begin{array}{c} \hat{\mathbf{P}}_L \\ \hat{\mathbf{P}}_R \end{array} \right) \right\|^2 =$$

$$= \left\| \mathbf{B} \left(\begin{array}{c} \mathbf{P}_L \\ \mathbf{P}_R \end{array} \right) - \left(\begin{array}{c} \hat{\mathbf{P}}_L \\ \hat{\mathbf{P}}_R \end{array} \right) \right\|^2$$

$$= \frac{1}{2} \left\| \left(\begin{array}{c} \mathbf{R}\mathbf{P}_L - \mathbf{R}\hat{\mathbf{P}}_L - \mathbf{P}_R + \hat{\mathbf{P}}_R \\ \mathbf{P}_L - \hat{\mathbf{P}}_L + \mathbf{R}\mathbf{P}_R - \mathbf{R}\hat{\mathbf{P}}_R \end{array} \right) \right\|^2$$

$$= \frac{1}{2} \left\| \left(\begin{array}{c} \mathbf{R}\mathbf{P}_L - \mathbf{P}_R \\ \mathbf{P}_L + \mathbf{R}\mathbf{P}_R - 2\hat{\mathbf{P}}_L \end{array} \right) \right\|^2$$

$$= \frac{1}{2} \left(\|\mathbf{R}\mathbf{P}_L - \mathbf{P}_R\|^2 + \left\| \mathbf{P}_L + \mathbf{R}\mathbf{P}_R - 2\hat{\mathbf{P}}_L \right\|^2 \right).$$

Thus the best symmetric approximation to the original data set is formed by

$$\mathbf{U}\mathbf{S}\mathbf{V}^T = \left(\begin{array}{c} \hat{\mathbf{P}}_L \\ \hat{\mathbf{P}}_R \end{array} \right)$$

where

$$\hat{\mathbf{P}}_L = \frac{1}{2}\left(\mathbf{P}_L + \mathbf{R}\mathbf{P}_R\right)$$

$$\hat{\mathbf{P}}_R = \mathbf{R}\hat{\mathbf{P}}_L.$$

Moreover, the best symmetric rank ℓ-approximation to $\mathbf{P}_L$ may be formed by the best rank-ℓ approximation to $\hat{\mathbf{P}}_L$. $\square$

Theorem 2.1 proves that the best symmetric low rank approximation to the face space is constructed by taking the SVD of the average of the first $\frac{mn}{2}$ rows of the training set $\mathbf{P}$ with the last $\frac{mn}{2}$ rows of the training set $\mathbf{P}$ in reverse order. In other words, if

$$\hat{\mathbf{P}}_L = \frac{1}{2}\left(\mathbf{P}_L + \mathbf{R}\mathbf{P}_R\right),$$

where $\mathbf{P}_L$ is the first $\frac{mn}{2}$ rows of the training set and $\mathbf{P}_R$ is the last $\frac{mn}{2}$ rows of the training set in reverse order, then an orthogonal basis for the best "symmetric face space" is calculated by the left singular vectors of the SVD of

$$\mathbf{U}_L\mathbf{S}_L\mathbf{V}_L^T = \hat{\mathbf{P}}_L = \frac{1}{2}(\mathbf{P}_L + \mathbf{R}\mathbf{P}_R).$$

Note, that the size of the left singular vectors of the symmetric face space are half the size of the left singular vectors of the conventional face space. This construction leads directly to an algorithm for facial recognition that is outlined in the next section.

3. Facial Recognition. The SVD and SPSVD can both be used to identify a new image from a training set of images. The idea is formulated on the PCA algorithm by Turk and Pentland [7]. The algorithm begins by calculating a basis for the (symmetric) face space from the SVD/SPSVD of a given training set of images as outlined in the previous section. A new image is identified from the training images by projecting the image onto the respective (symmetric) face space. In other words, a given image P is projected onto the rank ℓ (symmetric) face space by the operation

$$\mathbf{w}_\ell = \mathbf{U}_\ell^T(P - \Psi)$$

where Ψ is the average face of the training set and $\mathbf{U}_\ell$ are the first ℓ (symmetric) eigenfaces. This operation results in a vector of ℓ weights $\mathbf{w}_\ell$ that describes the influence of each eigenface on the given new image. Then a nearest-neighbor algorithm is used to identify the image from the training set. In other words, the weight vector of the new image is compared to the weight vector of each of the images in the training set. The closest image of the training set (in the Euclidean norm) is deemed the match. In other words, to identify a given face from a training set of images:

1. Create a basis for the symmetric/unsymmetric face space using the SPSVD/SVD.
2. Project a given image onto the symmetric/unsymmetric face space to calculate a weight vector.
3. Perform a nearest neighbor algorithm to identify the image from the training set.

FIG. 4.1. *Images from the trial set.*

SPSVD

1   2   3   4   5

SVD

1   2   3   4   5
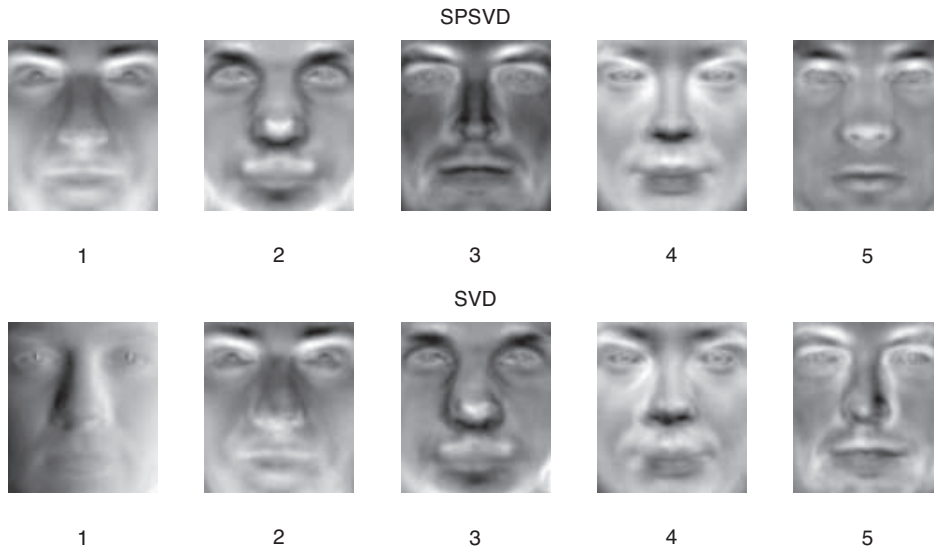
FIG. 4.2. *Symmetric verses Unsymmetric Eigenfaces*

4. Computational Results. The SPSVD and SVD were compared in a series of experiments with increasing light variations and facial occlusions on a dataset that consists of 618 images from the Harvard Robotics Laboratory (HRL dataset) [2] (downloaded from http://www1.cs.columbia.edu/∼belhumeur/ on 4/30/2008). The training set from these experiments consisted of 60 images of 10 distinct people of varying age, sex, and race. Examples of each of the 10 distinct people are shown in Figure 4.1. The experiments begin with calculating the (symmetric) eigenfaces of the base trial set. The first five (symmetric) eigenfaces are shown in Figure 4.2. Then two sets of images were projected onto these (symmetric) face spaces. The first set consisted of images with increasing light variations, and the second set consisted of images with increasing facial occlusions.

Results from experiments with increasing light variations are shown in Figure 4.3. A nice description of these light variations is discussed in [1]. In particular, the subjects are asked to keep their faces still while being illuminated by lighting sources parametrized by spherical angles in 15 degree increments ranging from 15 degrees up to 90 degrees. Figure 4.3 shows the results from these experiments. On the x-axis the rank of the (symmetric) face space is shown from rank 5 up to rank 30, where as the y-axis shows the percentage of correct matchings. It is clear that the SPSVD performs at least as well as the SVD in most of these cases. In addition, the SPSVD performs these calculations in half the time, since the symmetric face space is half the dimension of the unsymmetric face space.

Similar experiments with increasing facial occlusions are shown in Figure 6.1. In these experiments, portions of the face are occluded in 12.5% increments ranging from
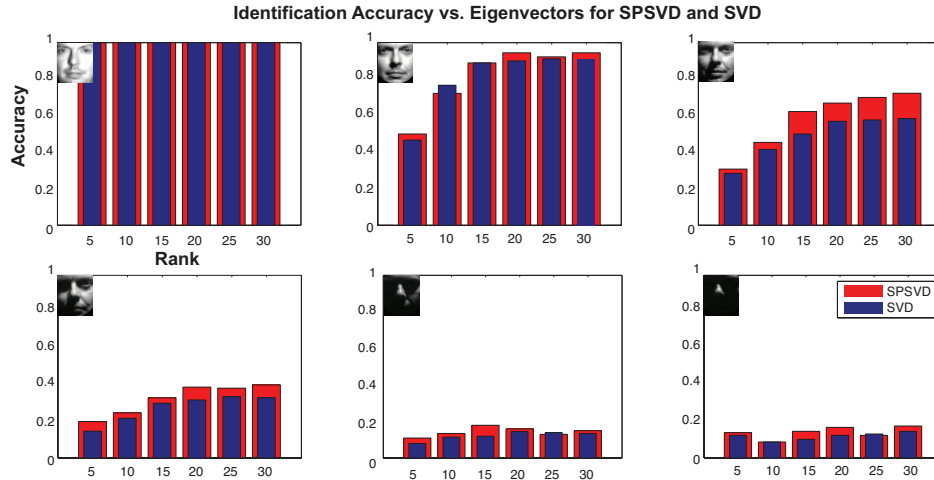
Fig. 4.3. *Percentage of correct identification between the SPSVD (red) and the SVD (blue) with increased lighting variation and increased dimensionality of the face database.*

12.5% up to 100%. On the x-axis the rank of the (symmetric) face space is shown from rank 5 up to rank 30, where as the y-axis shows the percentage of correct matchings. As in the light variation case, the SPSVD works at least as well as the SVD case in half the amount of computations. It should be noted that when the facial occlusions are around 50% the SPSVD has almost double the accuracy as the SVD case. This result should be obvious since the SPSVD averages out the occluded portion of the face.

5. Conclusion. This paper introduces a new facial recognition algorithm, the symmetry preserving singular value decomposition (SPSVD). The SPSVD takes advantage of the inherent symmetry of faces to increase the accuracy and efficiency of the SVD to match images from a given training set of images. The SPSVD is applied on experiments with increasing light variations and facial occlusions.

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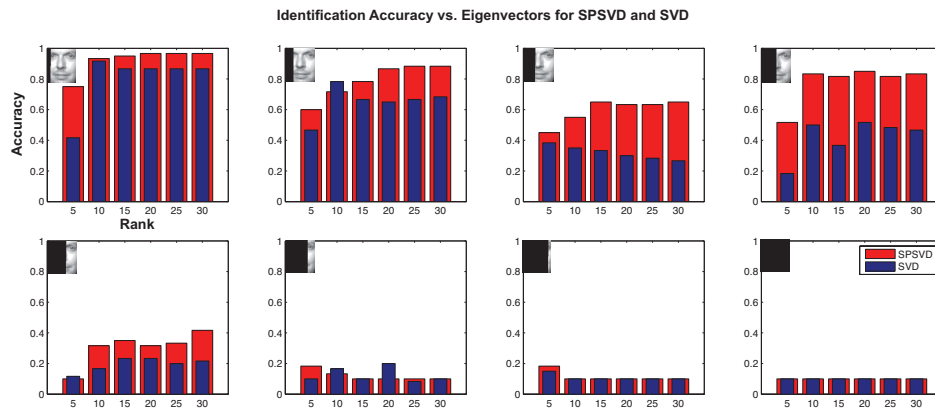
Fig. 6.1. *Percentage of correct identification between the SPSVD (red) and the SVD (blue) with increased facial occlusions and increased dimensionality of the face database.*

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