MA 428: Homework 2: Numerical differentiation (Solutions)

Theoretical

Section 6.2 – 4a,4b,12;

4 a To derive the difference approximation

$$f'(x_0) \approx \frac{f(x_0+2h) - f(x_0-h)}{3h}$$

We interpolate then differentiate the interpolant, so let $x_0 - h$ and $x_0 + 2h$ be the interpolating nodes. We write the lagrange form of the interpolant:

$$f(x) = -\frac{x - x_0 - 2h}{3h} f(x_0 - h) + \frac{x - x_0 + h}{3h} f(x_0 + 2h) + f[x_0 - h, x_0 + 2h, x](x - x_0 + h)(x - x_0 - 2h)$$

Now we differentiate

$$\begin{aligned} f'(x) &= -\frac{1}{3h} f(x_0 - h) + \frac{1}{3h} f(x_0 + 2h) \\ &+ (x - x_0 + h)(x - x_0 - 2h)\frac{d}{dx} f[x_0 - h, x_0 + 2h, x] \\ &+ [2(x - x_0) - h] f[x_0 - h, x_0 + 2h, x]. \end{aligned}$$

Evaluating at $x = x_0$:

$$f'(x_0) = \frac{f(x_0+2h) - f(x_0-h)}{3h} - hf[x_0 - h, x_0 + 2h, x_0] - 2h^2 \frac{d}{dx} f[x_0 - h, x_0 + 2h, x_0]\Big|_{x=x_0}$$

The first term on the right handside constitutes the finite difference approximation; thus,

$$f'(x_0) \approx \frac{f(x_0+2h) - f(x_0-h)}{3h}$$

b The error term is

$$-hf[x_0 - h, x_0 + 2h, x_0] = -\frac{h}{2} f''(\xi), \quad \text{for } \xi \in (x_0 - h, x_0 + 2h)$$

Note that we ignore the error term involving h^2 because for small values of h , $h^2 < h$.

12 For each value of f you should obtain the true value of the second derivative upto $f(x) = x^3$.
