Gaussian Quadrature

Megan Rosinsky

Loyola University Maryland

December 7, 2015

Outline

Gaussian Quadrature Overview

- A quadrature method to approximate the definite integral.
- The abscissas and weights are selected to achieve the highest possible degree of precision.
- Newtons-Cotes versus Gaussian quadrature
  - Newton-Cotes quadrature the nodes are evenly spaced over the interval of integration.
  - Gaussian quadrature the abscissas and weights are selected to achieve the highest possible degree of precision.

Method of Undetermined Coefficients

To develop the Gaussian Quadrature rule, one would use the method of undetermined coefficients.

- We will use the definition of degree of precision
  - Degree of Precision: *2n-1*
- Given a positive integer, n, we want to determine *2n* numbers
  - The abscissas: $x_1, x_2, \ldots, x_n$
  - The weights: $w_1, w_2, \ldots, w_n$
- The goal is to have the summation $w_1 f(x_1) + w_2 f(x_2) + \cdots + w_n f(x_n)$ equal to the exact value of $\int_a^b f(x)dx$ for $f(x) = 1, x, x^2, x^3, \ldots, x^{2n-1}$.

Gaussian Quadrature Rule

We will develop the Gaussian Quadrature rule for the n=1 case. For the n=1 case, we will have exact values for all constants and all linear functions. Thus we get the following system of equations:

$$f(x) = 1; w_1 = \int_a^b dx = b - a$$

$$f(x) = x; w_1 x_1 = \int_a^b x dx = \frac{1}{2}(b^2 - a^2)$$

The system of equations has a solution $w_1 = b - a$ and $w_1 x_1 = \frac{1}{2}(b^2 - a^2)$. Thus $w_1 = b - a$ and $x_1 = \frac{a+b}{2}$. Thus the Gaussian Quadrature for n=1 is

$$\int_a^b f(x)dx \approx (b - a)f(\frac{a+b}{2})$$

Gaussian Quadrature Rule

It is to our advantage to solve for the general rule with the standard interval [-1,1]. After the change of variable the resulting integral will be

$$\int_a^b f(x)dx = \frac{b-a}{2}\int_{-1}^1 f(\frac{b-a}{2}t + \frac{a+b}{2})dt.$$

Gaussian Quadrature Rule

We will develop the Gaussian Quadrature rule for the n=2 case. The degree of precision is now 3.The weights and abscissas must satisfy:

- $f(x) = 1 : w_1 + w_2 = \int_{-1}^{1} dx = 2$
- $f(x) = x : w_1x_1 + w_2x_2 = \int_{-1}^{1} xdx = 0$
- $f(x) = x^2 : w_1x_1^2 + w_2x_2^2 = \int_{-1}^{1} x^2dx = \frac{2}{3}$
- $f(x) = x^3 : w_1x_1^3 + w_2x_2^3 = int_{-1}^{1}x^3dx = 0$

We can see that $x_2 = -x_1$ and $w_1 = w_2$. Thus

$$\int_{-1}^{1} f(t)dt \approx f(\sqrt{\frac{1}{3}}) + f(-\sqrt{\frac{1}{3}})$$

Error Term

To find the error associated with the two point Gaussian quadrature rule, we will interpolate the integrand, f, at $x_1 = \sqrt{\frac{1}{3}}$ and $x_2 = -\sqrt{\frac{1}{3}}$. The formula associated with the two point Gaussian quadrature rule is $\int_{-1}^{1} f[x_1, x_2, x](x - x_1)(x - x_2)dx$. Thus the two point Gaussian quadrature rule is

$$\int_{-1}^{1} f(t)dt = f(\sqrt{\tfrac{1}{3}}) + f(-\sqrt{\tfrac{1}{3}}) + \tfrac{1}{135} f^{(4)}(\xi).$$

General Integration Interval

To change this rule from the standard interval of [-1,1] to a general interval of [a,b]:

$\int_a^b f(x)dx$
$= \frac{b-a}{2} \int_{-1}^{1} f(\frac{b-a}{2}t + \frac{a+b}{2})dt$
$= \frac{b-a}{2}[f(\frac{a+b}{2} + \sqrt{\frac{1}{3}}\frac{b-a}{2}) + f(\frac{a+b}{2} - \sqrt{\frac{1}{3}}\frac{b-a}{2}) + \frac{1}{135}\frac{d^4 f}{dt^4}(\xi)]$
$= \frac{b-a}{2}[f(\frac{a+b}{2} + \sqrt{\frac{1}{3}}\frac{b-a}{2}) + f(\frac{a+b}{2} - \sqrt{\frac{1}{3}}\frac{b-a}{2})] + \frac{(b-a)^5}{4320}\frac{d^4 f}{dt^4}(\xi)$

Example 6.14

Approximate the value of $ln(2)$
One way to approximate the value of $ln(2)$ is using the integral

$$\int_1^2 \frac{1}{x} dx.$$

We can use the two-point Gaussian quadrature rule. In this problem, a=1, b=2, and f(x)=$\frac{1}{x}$. Thus we have the approximation

$$\int_1^2 \frac{1}{x} dx \approx \frac{2-1}{2}[(\frac{2+1}{2}) - \sqrt{\frac{1}{3}}(\frac{2-1}{2})^{-1} + (\frac{2+1}{2}) + \sqrt{\frac{1}{3}}(\frac{2-1}{2})^{-1}]$$
$$= \frac{1}{2}[(\frac{3}{2} - \frac{\sqrt{3}}{6})^{-1} + \frac{3}{2} + \frac{\sqrt{3}}{6})^{-1}]$$
$$= 0.6923076923$$

The actual value of $ln(2)$ is 0.6931471805. Thus we get an error of $8.394x10^{-4}$.

Example 6.15

Approximate the value of π

- Using Simpson's rule with 12 subintervals we can estimate the integral $\int_0^1 \frac{1}{1+x^2} dx$
  - Value of the integral: 0.78539816007634
  - Multiply this number by 4 and we get an approximate value of $\approx 3.141592664030538$.
  - Error: $1.3284x10^{-8}$.
- Using two point Gaussian quadrature rule with n=5 subintervals we can estimate the integral $\int_0^1 \frac{1}{1+x^2} dx$
  - Value of the integral: 0.78539817044636
  - Multiply this number by 4 and we get an approximate value of ≈ 3.14159268178543.
  - Error: $2.8196x10^{-8}$

This error is larger than that found with Simpson's method but Gaussian quadrature uses fewer function evaluations. The Gaussian quadrature uses 10 function evaluations and the Simpson's method uses 13.