Math 351
FALL 2017
Exam 2
11/14/2017
Instructor: Dr. Prince Chidyagwai
Time Limit: 75 mins

Name (Print): SOLUTIONS

This exam contains 7 pages (including this cover page) and 10 problems including a 5 point *optional* bonus problem. Check to see if any pages are missing. Enter all requested information on the top of this page.
You may *not* use your books or notes on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive any credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- Use proper notation to distinguish between scalars and vectors.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	10	
2	5	
3	10	
4	10	
5	15	
6	10	
7	15	
8	10	
9	15	
10	5	
Total:	105	

1. A particle moves along a path with position vector $\mathbf{r}(t) = \langle \cos(4t), \sin(4t), 3t \rangle$ for $0 \le t \le 1$.

(a) (5 points) Find the velocity of the particle at $t = \frac{\pi}{4}$.

$$\vec{v}(t) = \vec{r}\,'(t) = \langle -4\sin(4t),\ 4\cos(4t),\ 3 \rangle$$

$$\vec{v}\left(\frac{\pi}{4}\right) = \vec{r}\,'\left(\frac{\pi}{4}\right) = \langle -4\sin(\pi),\ 4\cos(\pi),\ 3 \rangle$$

$$= \langle 0, -4, 3 \rangle.$$

(b) (5 points) Find the length of the curve traced by the particle for $0 \le t \le 1$.

$$\ell = \int_0^1 |\vec{r}\,'(t)|\, dt$$

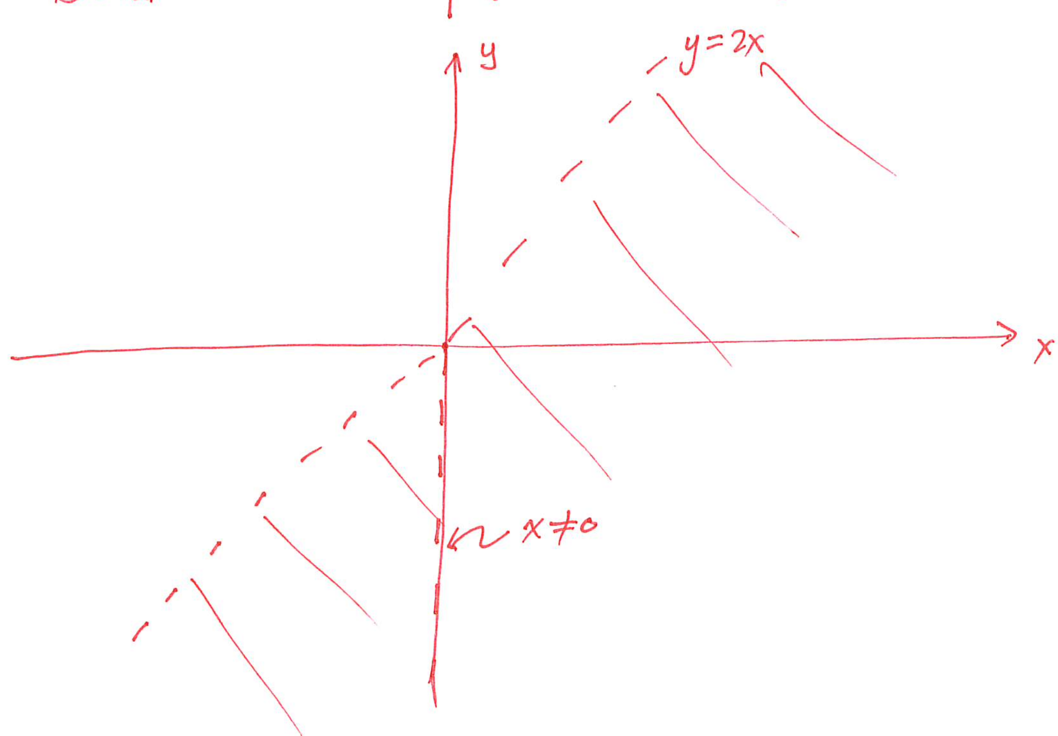
$$= \int_0^1 \sqrt{(-4\sin(4t))^2 + (4\cos(4t))^2 + 9}\, dt$$, note that $16\sin^2(4t) + 16\cos^2(4t) = \underline{16}$

$$= \int_0^1 \sqrt{16+9}\, dt = \int_0^1 5\, dt = 5t\Big|_0^1 = \underline{\underline{5}}$$

2. (5 points) Find and sketch the domain of $f(x,y) = \dfrac{\ln(2x-y)}{x}$

We need $2x - y > 0$ & $x \neq 0$

so Domain $= \{ (x,y) \mid y < 2x \ \&\ x \neq 0 \}$



3. Calculate the limit OR show that the limit does not exist

(a) (5 points) $\lim_{(x,y)\to(1,1)} \left(\frac{2xy}{x^2+2y^2}\right)$

$\frac{2xy}{x^2+2y^2}$ is continuous at $(1,1)$, therefore

$$\lim_{(x,y)\to(1,1)} \left(\frac{2xy}{x^2+y^2}\right) = \frac{2\cdot 1\cdot 1}{1^2+2\cdot(1)^2} = \frac{2}{3}.$$

(b) (5 points) $\lim_{(x,y)\to(0,0)} \left(\frac{2xy}{x^2+2y^2}\right)$

limit along x-axis (set $y=0$)

$$\lim_{x\to 0}\left(\frac{0}{x^2}\right) = 0$$

limit along y-axis

$$\lim_{y\to 0}\left(\frac{0}{2y^2}\right) = 0$$

so we check along $y=x$

$$\lim_{x\to 0}\left(\frac{2x^2}{x^2+2x^2}\right) = \lim_{x\to 0}\left(\frac{2x^2}{3x^2}\right) = \frac{2}{3}.$$

Therefore the limit D.N.E.

4. Find the first partial derivatives for each of the following

(a) (4 points) $f(x,y) = (5y^3+2x^2y)^8$.

Chain Rule!

$$f_x(x,y) = 8(5y^3+2x^2y)^7\cdot\frac{\partial}{\partial x}(5y^3+2x^2y) = 8(5y^3+2x^2y)\cdot 4xy$$

$$f_y(x,y) = 8(5y^3+2x^2y)^7\cdot\frac{\partial}{\partial y}(5y^3+2x^2y) = 8(5y^3+2x^2y)^7\cdot(15y^2+2x^2)$$

(b) (6 points) $G(x,y,z) = e^{xz}\sin\left(\frac{y}{z}\right)$.

$$G_x(x,y,z) = ze^{xz}\sin\left(\frac{y}{z}\right)$$

$$G_y(x,y,z) = e^{xz}\cos\left(\frac{y}{z}\right)\cdot\frac{1}{z}$$

$$G_z(x,y,z) = xe^{xz}\sin\left(\frac{y}{z}\right) + e^{xz}\cdot\cos\left(\frac{y}{z}\right)\cdot(-yz^{-2})$$

(product Rule)

5. (15 points) Find and classify the critical points of $f(x,y) = (x^2+y^2)e^{-x}$

First find the critical pts.

$$f_x(x,y) = (x^2+y^2)e^{-x}(-1) + e^{-x}\cdot 2x. \qquad f_y(x,y) = 2ye^{-x}$$
$$= e^{-x}(-x^2-y^2+2x)$$

$$f_x(x,y) = 0 \cdots (i) \Rightarrow e^{-x}(-x^2-y^2+2x) = 0$$
$$f_y(x,y) = 0 \cdots (ii) \Rightarrow 2ye^{-x} = 0.$$

From (ii), we have $2ye^{-x}=0 \Rightarrow y=0$ because $e^{-x} \neq 0$, then plug into (i) to obtain $e^{-x}(-x^2+2x) = 0 \Rightarrow -x^2+2x=0$ so $x=0$ or $x=2$

so we have 2 critical points $(0,0)$ and $(2,0)$.

Classification $D = f_{xx}\cdot f_{yy} - (f_{xy})^2$. so we compute

$$f_{xx} = e^{-x}(-2x+2) + (-x^2-y^2+2x)(-1)e^{-x} = e^{-x}(-4x+2+x^2+y^2)$$
$$f_{yy} = 2e^{-x}, \quad f_{xy} = f_{yx} = -2ye^{-x}.$$

Critical points	f_{xx}	f_{yy}	f_{xy}	D	Classification
$(0,0)$	2	2	0	4	local min
$(2,0)$	$-2e^{-2}$	$2e^{-2}$	0	$-4e^{-4}$	Saddle.

6. (10 points) Find an equation for the tangent plane to the surface $4x^2+9y^2-z^2 = 16$ at $P = (2,1,3)$.

The equation of the tangent plane is

$$F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0) = 0$$

$$\nabla F = \langle 8x, 18y, -2z \rangle$$

$\nabla F(2,1,3) = \langle 16, 18, -6 \rangle$ so the equation of the Tangent Plane is

$$\boxed{16(x-2) + 18(y-1) - 6(z-3) = 0}$$

7. A lonely cold bug located on a surface at $(1,1,1)$ begins moving towards $(2,0,2)$. The temperature of the surface in $°C$ is given by $T(x,y,z) = xe^{y-z}$ and the position is in centimeters.

   (a) (10 points) Find the rate of change of the temperature at $(1,1,1)$ in the direction of movement of the bug.

$$D_u T(P) = \nabla T(P) \cdot \vec{u}.$$

$$\nabla T = \langle e^{y-z}, xe^{y-z}, -xe^{y-z} \rangle \quad \text{so } \nabla T(1,1,1) = \langle 1, 1, -1 \rangle$$

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|}, \text{ where } \vec{v} = \langle 2-1, 0-1, 2-1 \rangle = \langle +1, -1, 1 \rangle$$

$$\text{so } \vec{u} = \frac{1}{\sqrt{3}} \langle +1, -1, 1 \rangle \text{ therefore } D_{\vec{u}} T(1,1,1) = \langle 1, 1, -1 \rangle \cdot \frac{1}{\sqrt{3}} \langle 1, -1, 1 \rangle$$

$$= \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}} = -\frac{1}{\sqrt{3}} \ °C/cm.$$

   (b) (5 points) The bug wants to move to a warmer position at the quickest possible rate. Is the bug moving in the right direction? Explain.

No.

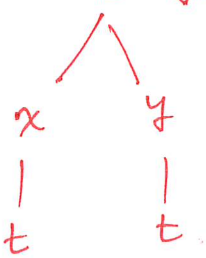
The bug needs to move in the direction $\nabla T(1,1,1) = \langle 1, 1, -1 \rangle$.

8. (10 points) If $z = f(x,y)$, were f is differentiable and

$$x = g(t) \quad y = h(t)$$
$$g(3) = 2 \quad h(3) = 7$$
$$g'(3) = 5 \quad h'(3) = -4$$
$$f_x(2,7) = 6 \quad f_y(2,7) = -8$$

find $\dfrac{dz}{dt}$ when $t = 3$.

$z = f(x,y)$

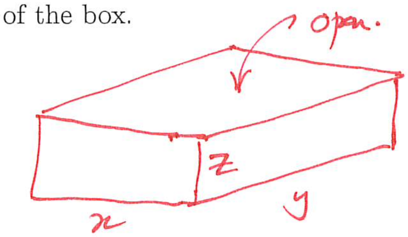


$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$$

using the fact that $\dfrac{dx}{dt} = g'(t)$ $\dfrac{dy}{dt} = h'(t)$. and $g(3) = 2$, $h(3) = 7$

$$\left.\frac{dz}{dt}\right|_{t=3} = f_x(2,7)\, g'(3) + f_y(2,7)\, h'(3)$$
$$6 \cdot 5 + (-8)(-4) = 62$$

9. (15 points) A manufacturer wants to make an open rectangular box of volume $V = 500\text{cm}^3$ using the least possible amount of material. Find the dimensions of the box.



$V = 500 = xyz. \Rightarrow \frac{500}{xy} = z$

Surface area

$2xz + 2yz + xy$

Next we write the surface area as a function of 2 variables (x and y)

$$f(x,y) = 2x\left(\frac{500}{xy}\right) + 2y\left(\frac{500}{xy}\right) + xy = \frac{1000}{y} + \frac{1000}{x} + xy$$

The objective is to find the critical points of $f(x,y)$.

$$f_x(x,y) = \frac{-1000}{x^2} + y \quad , \quad f_y(x,y) = \frac{-1000}{y^2} + x$$

$-\frac{1000}{x^2} + y = 0 \quad \ldots \text{(i)}$

$-\frac{1000}{y^2} + x = 0 \quad \ldots \text{(ii)}$

$\Rightarrow$ from (i) $y = \frac{1000}{x^2}$, plugging into (ii)

$-1000 \cdot \left(\frac{x^4}{1000^2}\right) + x = 0$

$\frac{-x^4}{1000} + x = 0$, so

$1000x = \frac{x^4}{\cancel{1000}} \cdot \cancel{1000}$

$1000x = x^4 \Rightarrow x^3 = 1000$ assume $x \neq 0$

so $x = 10$

From $y = \frac{1000}{x^2} = \frac{1000}{10^2} = 10$

$z = \frac{500}{10 \cdot 10} = 5$

Finally, we check that $(10,10)$ is a local min.

$f_{xx} = \frac{2000}{x^3} \qquad f_{yy} = \frac{2000}{y^3}, \quad f_{xy} = 1 \quad$ so

$$D(10,10) = f_{xx}(10,10) \cdot f_{yy}(10,10) - 1^2$$

$$= \left(\frac{2000}{10^3}\right) \cdot \left(\frac{2000}{10^3}\right) - 1^2 = 2 \cdot 2 - 1 > 0,$$

Note that $f_{xx} > 0$ so $(10,10)$ is a local min. The dimensions of the box are $10 \times 10 \times 5$

10. (5 points) (BONUS) For each of the following determine whether the statement is **TRUE** or **FALSE**. To receive any credit, you must fully justify your solution.

(a) If $f(x,y) \to L$ as $(x,y) \to (0,0)$ along every straight line through $(0,0)$, then $\lim_{(x,y)\to(0,0)} f(x,y) = L$.

False

From class, we know that one has to check every possible curve to $(0,0)$

(b) If $f(x,y) = \ln y$, then $\nabla f(x,y) = \frac{1}{y}$

False $\nabla F(x,y) = \langle 0, \frac{1}{y} \rangle$

(c) If C is the curve of intersection of the surfaces $F(x,y,z) = 0$ and $G(x,y,z) = 0$ and $P = (x_0, y_0, z_0)$ is a point on C then the direction vector of the tangent line to C is given by $v = \nabla F(x_0, y_0, z_0) \times \nabla G(x_0, y_0, z_0)$.

the ∇F is perpendicular to C

∇G is also perpendicular to C,

therefore the direction vector of C is $\nabla F \times \nabla G$