MA 428: Homework 2: Numerical Integration

Due: Wednesday, October 4

Theoretical

1. Prove that if an integration formula of the form $I_n(f) = \sum_{i=1}^{n} w_i f(x_i)$ is exact when integrating $1, x, x^2, \ldots, x^m$, then it is exact for all polynomials of degree $\leq m$.
2. Problem 1 on page 248 (HINT: Use the method of undetermined coefficients)
3. Problem 3 on page 249.
4. Problem 6 on page 249.
5. Determine the smallest value of n which guarantees that the composite trapeziodal rule approximates the value of $\int_0^1 \frac{4}{1+x^2}\, dx$ to within 1×10^{-5}

Computational

Composite Quadrature

In this assignment you will numerically verify the convergence of the composite Trapezoidal rule and implement the composite Simpsons rule.

Please note that to numerically verify the rate of convergence of the method you need to give a table with at least 6 values showing the decay of the error as h decreases

Composite Trapezoidal Rule:

We divide the interval of integration $[a, b]$ into n subintervals with $h = \frac{(b-a)}{n}$ and $x_j = a + (j-1)h, 1 \leq j \leq n+1$. Applying the Trapezoidal rule on each subinterval, we obtain

$$\int_a^b f(x)\, dx = \frac{h}{2}\big[f(x_1) + 2\sum_{j=2}^{n} f(x_j) + f(x_{n+1})\big] - \frac{(b-a)}{12} h^2 f''(\eta)$$

where $\eta \in [a, b]$.

Composite Simpsons' Rule:

Since the basic Simpson's rule formula divides the interval $[a, b]$ into two pieces we must divide the interval $[a, b]$ into an even number of subintervals ($n = 2m$) to apply Simpsons rule in a composite manner m times, once over each subinterval $[x_{2j-1}, x_{2j+1}]$ for $j = 1, 2, 3, \cdots, m$. This results in the following composite Simposon's rule:

$$\int_a^b f(x)\, dx = \frac{h}{3}\big[f(x_1) + 4\sum_{j=1}^{m} f(x_{2j}) + 2\sum_{j=1}^{m-1} f(x_{2j+1}) + f(x_{2m+1})\big] - \frac{(b-a)h^4}{180} f^{(4)}(\eta).$$

where $\eta \in [a, b]$.

I have provided a matlab function:

```
function approx_int = composite_quadrature(f,a,b,n,rule)
```

that takes as input the function f to be integrated on the interval $[a, b]$ with n subintervals using the Composite Trapezoidal rule (rule A) and the Composite Simpson's rule (rule B) that you will implement. To verify convergence use the attached script `run_composite_rule.m` to generate a table of approximate integrals for increasing n, the errors and the ratio of the errors.
To compute the error, you will need the true value of the integral. You can either compute this by hand (when you can!) or use the Matlab function quad with the following usage

```
>> q=quad('f',0,pi,[1.e-12 1.e-12])
```

The above code will compute $\int_0^{\pi} f(x)\, dx$ for a function f defined in a Matlab m-file `f.m` accurate to 10^{-12}.

Problems

1. (a) By making appropriate adjustments to the provided code, verify that the composite trapezoidal rule converges with rate second order in h. by approximating the integral

$$\int_0^1 \frac{4}{1+x^2}\, dx$$

   (b) Verify that the value of n you computed in Problem 5 yeilds an error less than 1×10^{-5}

2. Implement the composite Simpsons rule inside the provided `composite_quadrature` function.

3. What is the rate of convergence of the composite Simpsons' method?

4. Numerically verify the rate of convergence for the composite Simpsons rule for the integrals

   A. $\int_0^1 \frac{4}{1+x^2}\, dx$

   B. $\int_0^1 cos(x^2)\, dx$

5. For the Integral A (above) compare and contrast your errors from the composite trapeziodal method in Problem 1 to the composite Simpsons method in Problem 4. Which method do you prefer? Why?

Submission

Email your zipped m files, including your **summary file** with a discussion of your results to `pchidyagwai@loyola.edu` with email heading MA428_HWn, where n is the assignment number. Your summary file must include all matlab output and answers to questions related to the output.