MA 428: Homework 4: High order methods

Due: Friday, November 10

Theoretical

1. Write down the result of applying one step of the midpoint and Heun's methods using step size $h = 0.1$ to the following equations

   (a)
   $$y'(t) = (t+1)e^{y(t)}, \quad y(0) = 0$$

   (b)
   $$\begin{aligned} R'(t) &= (2 - F(t))R(t), & (1) \\ F'(t) &= (R(t) - 2)F(t) & (2) \end{aligned}$$

   starting with $R_0 = 2$ and $F_0 = 1$.

2. Identity each of the following as representing a one-step method or a multi-step method (if multi-step, state the number of steps) and as being implicit or explicit

   (a) $\dfrac{y_{k+1} - y_k}{h} = \dfrac{3}{2}f(t_k, y_k) - \dfrac{1}{2}f(t_{k-1}, y_{k-1})$

   (b) $\dfrac{y_{k+1} - y_k}{h} = f\big(t_k + \dfrac{h}{2}, y_k + \dfrac{h}{2}f(t_k, y_k)\big)$

   (c) $\dfrac{y_{k+1} - y_k}{h} = \dfrac{5}{12}f(t_{k+1}, y_{k+1}) + \dfrac{2}{3}f(t_k, y_k) - \dfrac{1}{12}f(t_{k-1}, y_{k-1})$

   (d) $\dfrac{y_{k+1} - 4y_k + 3y_{k-1}}{h} = -2f(t_{k-1}, y_{k-1})$

   (e) $\dfrac{y_{k+1} - \frac{1}{2}y_k - \frac{1}{2}y_{k-1}}{h} = f(t_{k+1}, y_{k+1}) - \dfrac{1}{4}f(t_k, y_k) + \dfrac{3}{4}f(t_{k-1}, y_{k-1})$

3. Derive the 3 step Adams-Bashforth method
   $$\frac{y_{k+1} - y_k}{h} = \frac{23}{12}f(t_k, y_k) - \frac{4}{3}f(t_{k-1}, y_{k-1}) + \frac{5}{12}f(t_{k-2}, y_{k-2})$$

4. Problem 9 on page 296.

5. Problem 11 on page 296.

Computational

1. The 4^{th} order Runge-Kutta ($RK4$) method updates the numerical solution according to the formula
   $$y_{k+1} = y_k + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

   where

   $$\begin{aligned} k_1 &= hf(t_k, y_k) \\ k_2 &= hf(t_k + \frac{h}{2}, y_k + \frac{k_1}{2}) \\ k_3 &= hf(t_k + \frac{h}{2}, y_k + \frac{k_2}{2}) \\ k_4 &= hf(t_k + h, y_k + k_3) \end{aligned}$$

(a) Write a function `approx_ode.m` that implements the ($RK4$) method with the following heading

```
function [global_error,t_vals,approx_sol] = approx_ode(f,a,b,initial_val,N)
```

where the function `f.m` contains the definition of the initial value problem. Your function should return the global error at `t=b`, `t_vals` (a vector of $t_k, k = 0, 1, \ldots, N$) and `approx_sol` (a vector containing the approximate solution $y_k, k = 0, 1, \ldots N$).

(b) Consider the initial value problem

$$\frac{dy}{dt} = -(1 + t + t^2) - (2t + 1)y - y^2, \quad (0 \leq t \leq 3), \quad y(0) = -\frac{1}{2}$$

The exact solution of this problem is $y(t) = -t - \dfrac{1}{e^t + 1}$. Using your code, run $RK4$ to march from `y = initial_val` for each of the number of steps `N` in the table below. In addition compute the error as the absolute value of the difference between your approximate solution and exact solution at `t=3` and compute the ratios between successive errors. The first line is meant as a guide to the expected format and to check the correctness of your code.

RK4 method

N	Stepsize	Euler sol	Error	Ratio
10	0.3	-3.0474300	4.164107e-06	
20	0.15	----------	----------	---------
40	0.075	----------	----------	---------
80	0.0375	----------	----------	---------
160	0.01875	----------	----------	---------
320	0.009375	----------	----------	---------

2. A *genetic switch* is a biochemical mechanism that governs whether a particular protein product of a cell is synthesized or not. The following initial value problem is a model for a genetic switch:

$$\frac{dg}{dt} = s - 1.51g + 3.03\frac{g^2}{1 + g^2}, \quad g(0) = 0.$$

where g denotes the concentration of protein product and the parameter s denotes the concentration of chemical that activates the gene to produce the protein. The genetic model exhibits a *threshold effect* – this means that must exist a critical, or threshold value of the parameter s such that for values below the threshold the equilibrium concentration of the gene g stays near zero, but for values of s above this value the equilibrium gene concentration jumps to a higher value. You will use your $RK4$ code to examine the *threshold* effect by doing the following

(a) Run your code for $0 \leq t \leq 100$ for values of $s = 0.1, 0.2, 0.3, 0.4$ with $h = 0.2$ and plot your solutions on the same figure. Comment on your results, in particular state the equilibrium concentration for each value of s and determine the interval of the parameter s where the equilibrium concentration jumps.

(b) Once your have identified this interval perform more tests to further narrow it down. Say you interval is $[a, b]$ run your code for values of $s = a, a + 0.02, a + 0.04, a + 0.06$. Plot your results on the same graph, at this stage you should be able to narrow down the interval where the jump in equilibrium concentration occurs.

Submission

Email me your zipped m files, including your summary file with a discussion of your results for the computational part of the assignment. Your summary file must include all matlab output and answers to questions related to the output.