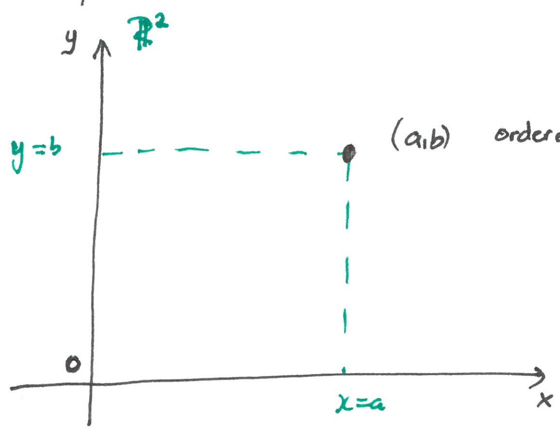
Three dimensional coordinate system

KEY ideas

1. Points in 3d
2. Equations of planes parallel to one coordinate system
3. Equation of sphere.

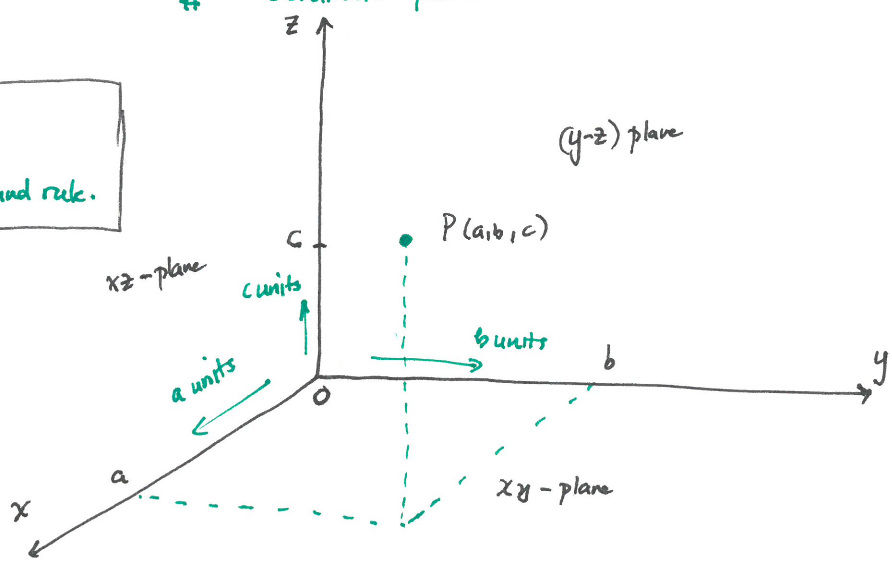
3 Dimensional coordinate system

In 2D, to locate a point we need ~~2 points~~ 2 real numbers to locate a point.



In 3 D space, we also need to start with an origin O
$\mathbb{R}^3$ coordinate planes

Note:
Right hand rule.



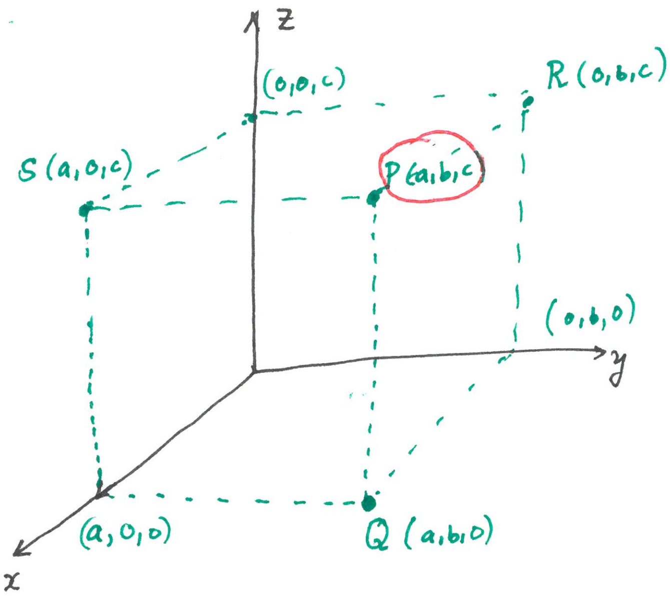
Use class room analogy

- 3 coordinate axes - x-axis, y-axis and z-axis
- The 3 coordinate systems define coordinate planes

To locate a point in 3D we need an ordered triple (a,b,c) of real numbers

Projections

Starting with a point $P(a,b,c)$ Use same Picture.



- If we drop $P(a,b,c)$ to the xy plane we get a point $Q(a,b,0)$.
- Q is called a projection of P onto the xy plane.
- S is the projection of P onto the $(x-z)$ plane.
- R is the projection of P onto the yz plane.

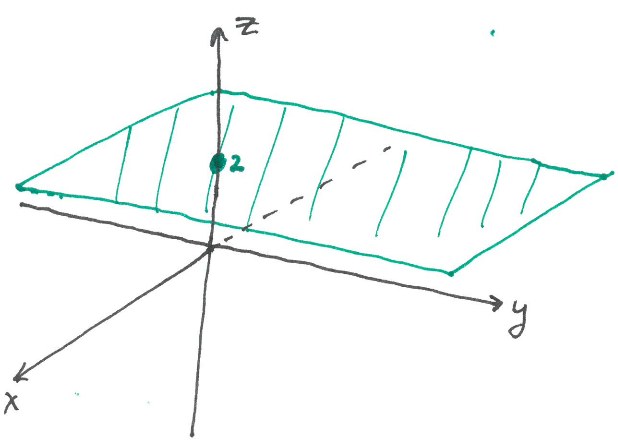
Notation

$\mathbb{R}^3 = \{(x,y,z) \mid x,y,z \in \mathbb{R}\}$ - Rectangular coordinate system.

Remark 1. In $\mathbb{R}^2$ (2D geometry), the graph of an equation $y=f(x)$ is a line. In $\mathbb{R}^3$ an equation in x, y, and z represents a surface in $\mathbb{R}^3$.

Planes parallel to the coordinate system

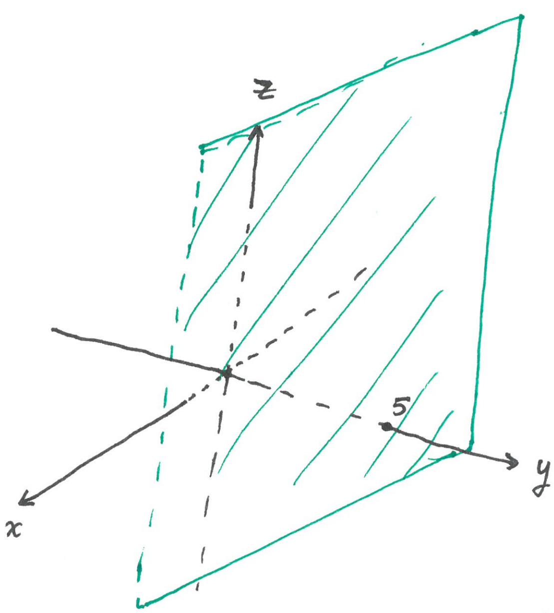
(i)



$z = 3$ in $\mathbb{R}^3 = \{(x,y,z) \mid z=3\}$

In general any plane parallel to the xy plane has equation $z = k$.

(ii)



$y = 5$ in $\mathbb{R}^3 = \{(x,y,z) \mid y=5\}$

In general any plane parallel to the xz plane has the form $y=k$

* In $\mathbb{R}^2$, these would be a line.

(iii) 12.1 In general $x = k$ represents a plane parallel to the yz plane.

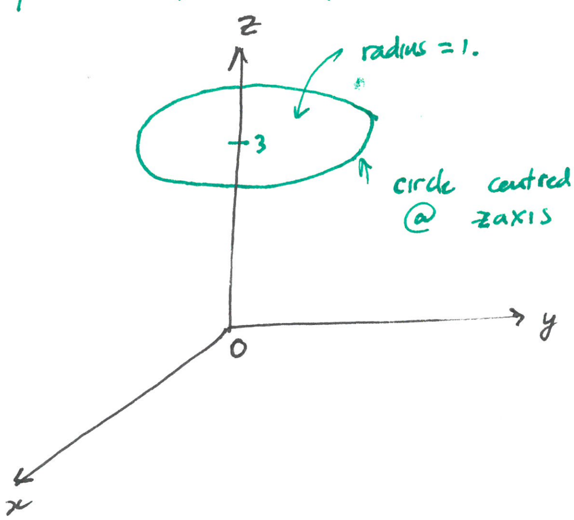
Equations of the coordinate planes

xy plane — $z = 0$
xz plane — $y = 0$
yz plane — $x = 0$.

Examples

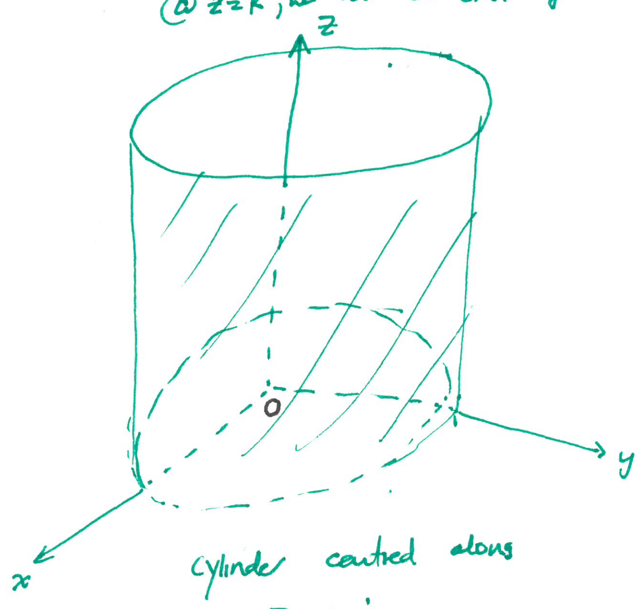
(a) $x^2 + y^2 = 1$ and $z = 3$

Which points (x, y, z) satisfy these above equations.



(b) $x^2 + y^2 = 1$ [No restrictions on z].

@ $z = k$, we have a circle of radius 1.



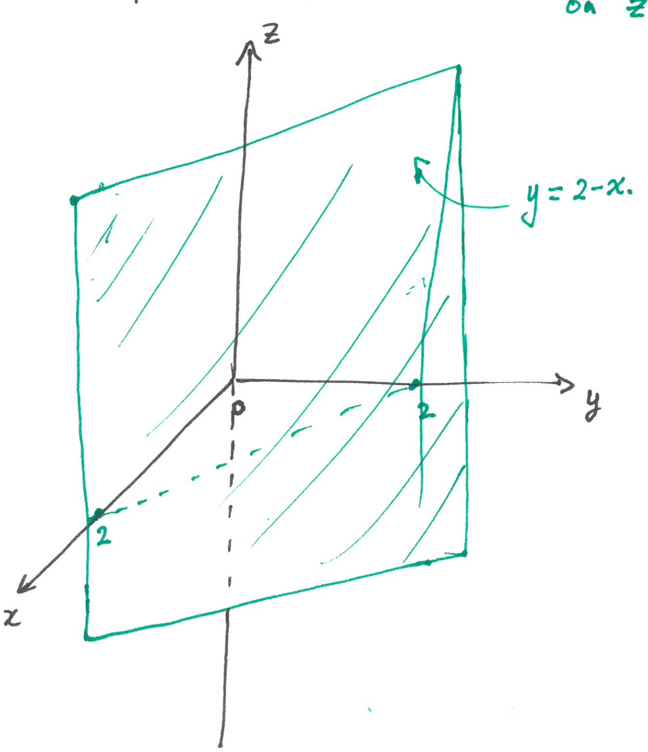
Cylinder centred along the z-axis.

$x^2 + y^2 = 1$ represents all circles of radius one along the z-axis.

Discussion

What about $x^2 + y^2 \leq 1$?

surface in $\mathbb{R}^3$ represented by $x + y = 2$. [no restriction on z]



$x + y = 2$ — all points in $\mathbb{R}^3$ whose x and y coordinates have a sum of 2.

$y = 2 - x$.

$\{(x, 2-x, z) \mid x \in \mathbb{R}, z \in \mathbb{R}\}$

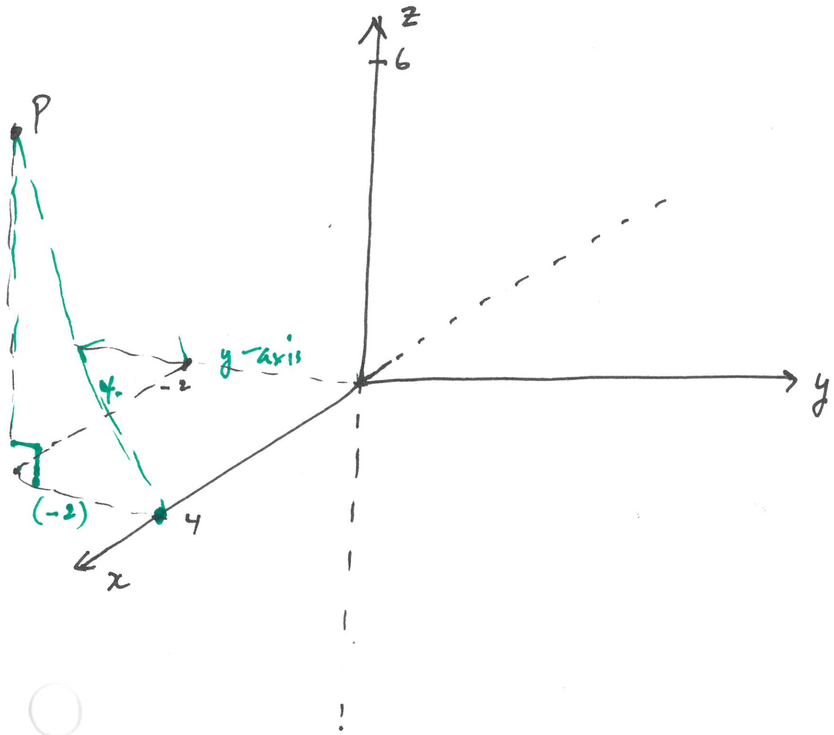
This is a vertical plane.

- Vertical plane intersecting the xy plane in the line $y = 2 - x$, $z = 0$.

12.1 Worksheet problems / Homework

- Find the distance from $(4,-2,6)$ to the coordinate planes.

- distance to xz-plane.



distance from P to the x-axis

$P = (4,-2,6)$.

The point closest to the x-axis is $(4,0,0)$.

$$\text{distance} = \sqrt{(4-4)^2+(-2)^2+(6)^2}$$
$$= \sqrt{40} \approx \underline{6.32}$$

Worksheet

$x^2+y^2 = z$ (surface)

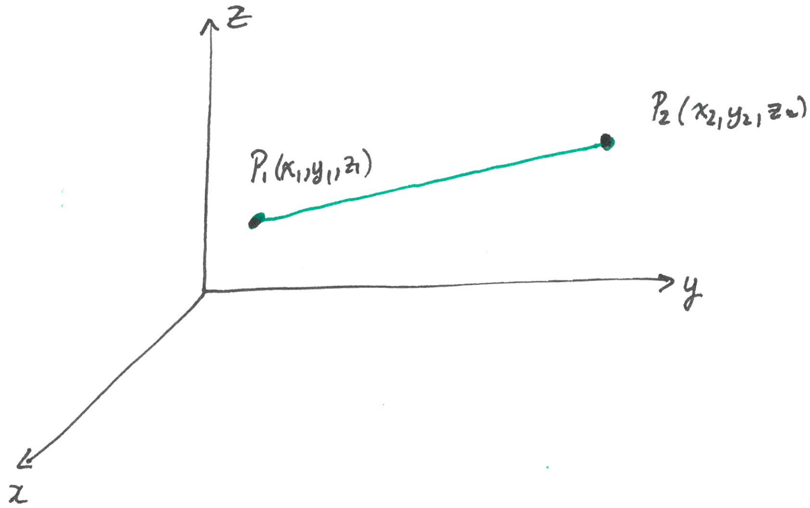
$x^2+y^2 = 3^2$

$z = y^2$.

12.1 Distance formula in 3D

The distance $|P_1P_2|$ between $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

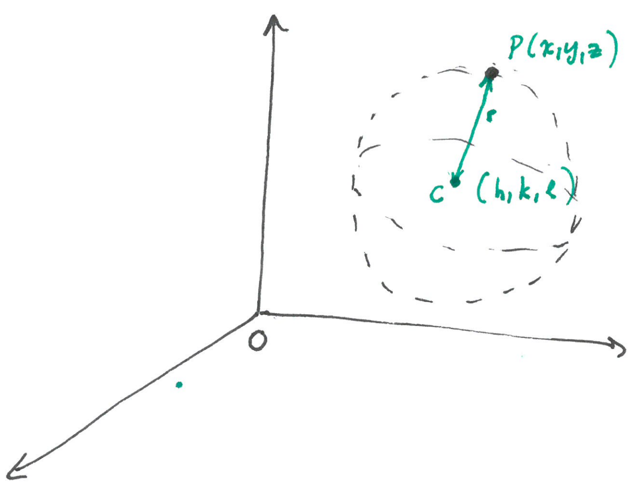
$$|P_1P_2| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$



Equation of a sphere

Sphere - definition

A sphere is the set of all points $P(x, y, z)$ whose distance is r.



$$\sqrt{(x-h)^2 + (y-k)^2 + (z-l)^2} = r$$

$$\boxed{(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2}$$

Is the equation of a sphere of radius r with centre (h, k, l).

Example

Find the equation of the sphere that passes through the point $(4, 3, -1)$ and center $(3, 8, 1)$.

$$\text{radius of sphere} = \sqrt{(4-3)^2 + (3-8)^2 + (-1-1)^2}$$

$$= \sqrt{30}$$

Equation of sphere is $(x-3)^2 + (y-8)^2 + (z-1)^2 = 30$.

Show that $x^2+y^2+z^2-2x-4y+8z=15$

represents the equation of a sphere. Find the radius and center.

$$x^2-2x(+1-1) + y^2-4y(+4-4) + z^2+8z(+16-16) = 15$$

$$(x-1)^2-1 + (y-2)^2-4 + (z+4)^2-16 = 15$$

$$(x-1)^2+(y-2)^2+(z+4)^2 = 15+1+4+16 = 36 = 6^2.$$

[complete the square]

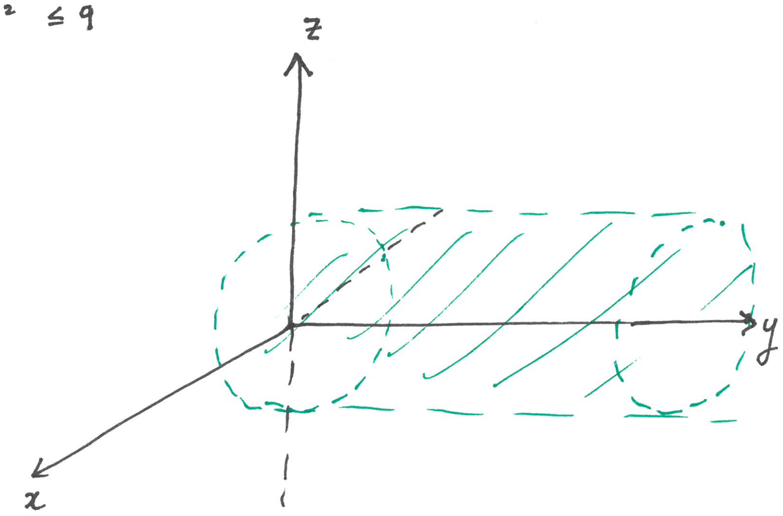
Center is $(x=1, y=2, z=-4)$ and Radius $=6$.

~~Descri~~

Regions in $\mathbb{R}^3$

(i) $0 \le z \le 6$ — All points between the horizontal planes $z=0$ and $z=6$.

(ii) $x^2+z^2 \le 9$



No restrictions on y.

All points on and inside a cylinder of radius 3 with centre along y-axis.

Midpoint of segment

The midpoint of a segment from $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$ is

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$$