12.3

THE DOT PRODUCT

Definition

If $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$, then the dot product of $\vec{a}$ and $\vec{b}$ is given by

$$\vec{a}\cdot\vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 .$$

Also known as the scalar product or inner product.

Properties of the Dot product

1. $\vec{a}\cdot\vec{a} = |\vec{a}|^2$ Notice that if $\vec{a} = \langle a_1, a_2, a_3 \rangle$

$$\vec{a}\cdot\vec{a} = \langle a_1, a_2, a_3\rangle \cdot \langle a_1, a_2, a_3\rangle$$
$$= a_1^2 + a_2^2 + a_3^2 = |\vec{a}|^2$$

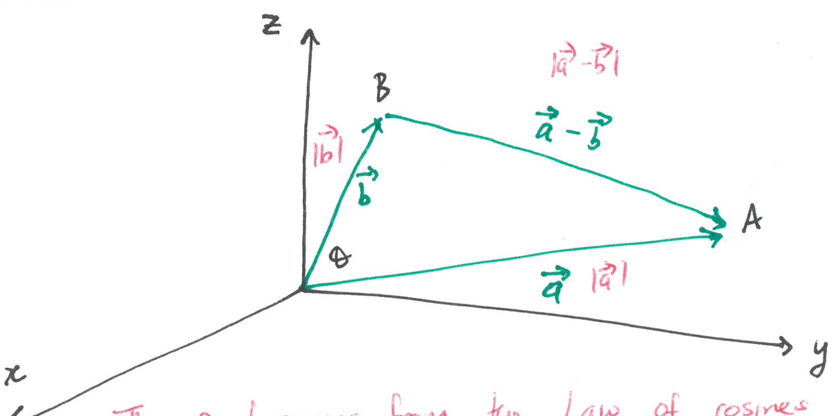
2. $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$

3. $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$

4. $(c\vec{a})\cdot\vec{b} = c(\vec{a}\cdot\vec{b}) = \vec{a}(c\vec{b})$.

5. $\vec{0}\cdot\vec{a} = 0$.

Geometric interpretation of dot product



~~Theo~~ FACT

If θ is the angle between $\vec{a}$ and $\vec{b}$, then

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

$$\Rightarrow \cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}.$$

The proof comes from the Law of cosines

$$|a-b|^2 = |a|^2 + |b|^2 - 2|a||b|\cos\theta$$

$$|\vec{a}-b|^2 = (\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b})$$

$$= \vec{a}\cdot\vec{a} - \vec{a}\cdot\vec{b} - \vec{b}\cdot\vec{a} + \vec{b}\cdot\vec{b}$$

$$= |\vec{a}|^2 - 2\vec{a}\cdot\vec{b} + |\vec{b}|^2$$

$$|\vec{a}|^2 - 2\vec{a}\cdot\vec{b} + |b|^2 = |a|^2 + |b|^2 - 2|a||b|\cos\theta$$

$$-2\vec{a}\cdot\vec{b} = 2|a||b|\cos\theta$$

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$

Example

Find the angle between ~~$\vec{a} = \langle 2,3,-1 \rangle$ and $\vec{b}$~~

$\vec{a} = 4i - 3j + k$

$\vec{b} = 2i - k$

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{\langle 4,-3,1\rangle\cdot\langle 2,0,-1\rangle}{|\vec{a}|\cdot|\vec{b}|} = \frac{4\cdot 2 + (-3)\cdot 0 + 1\cdot(-1)}{\sqrt{26}\cdot\sqrt{5}}$$

$$|\vec{a}| = \sqrt{4^2 + (-3)^2 + 1^2} = \sqrt{26}$$

$$= \frac{8 - 0 + (-1)}{\sqrt{26}\cdot\sqrt{5}} = \frac{7}{\sqrt{26}\cdot\sqrt{5}}$$

$$|b| = \sqrt{2^2 + (-1)^2} = \sqrt{5}$$

$$\cos\theta = \frac{7}{\sqrt{130}} \Rightarrow \theta = \cos^{-1}\left(\frac{7}{\sqrt{130}}\right).$$

~~Parallel~~

Perpendicular vectors

If the angle between two nonzero vectors $\vec{a}$ and $\vec{b}$ is θ ~~is given by~~

Since

$$\cancel{\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}.}$$

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cdot\cos(\theta)$$

If $\vec{a}$ and $\vec{b}$ are perpendicular / orthogonal then $\theta = \pi/2$

so that $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\left(\frac{\pi}{2}\right) = 0.$

Conversely, if $\vec{a}\cdot\vec{b} = 0$, then

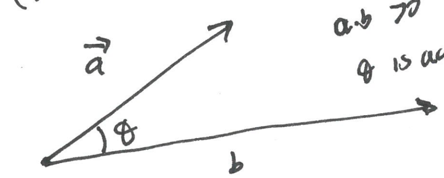
$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = 0 \quad \text{so} \quad \theta = \pi/2$$

We can conclude

Two non-zero vectors $\vec{a}$ and $\vec{b}$ are orthogonal if and only if $\vec{a}\cdot\vec{b} = 0$.

Geometric intuition of dot product

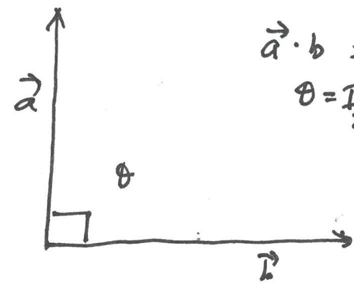
(i)



$a \cdot b > 0$

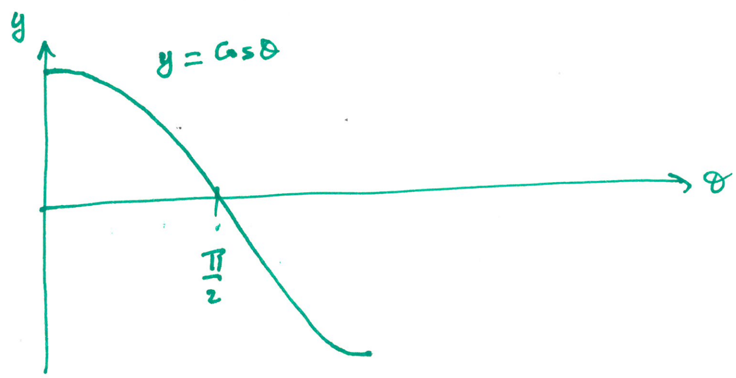
θ is acute

(ii)



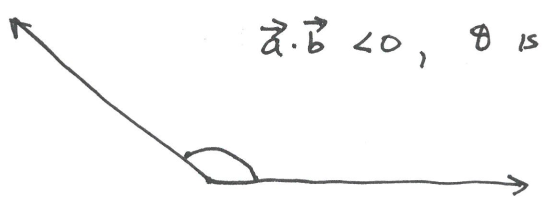
$\vec{a}\cdot b = 0$

$\theta = \frac{\pi}{2}$



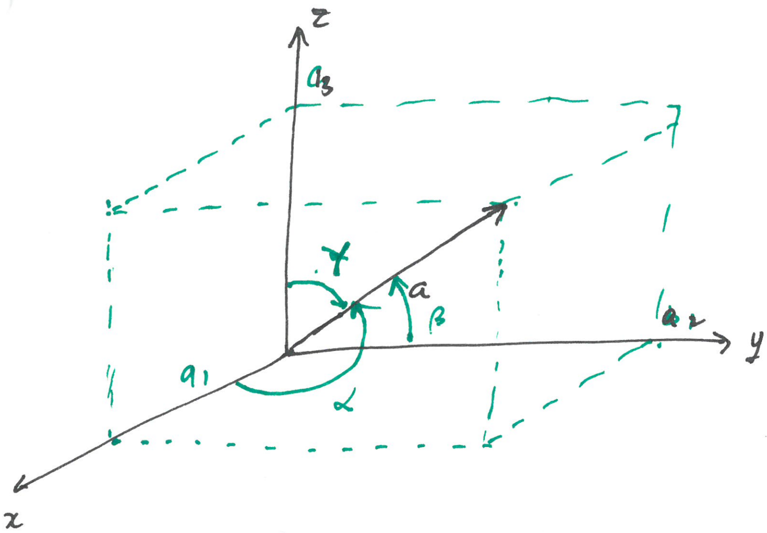
$\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta \Rightarrow$ The sign of $\vec{a}\cdot\vec{b}$ is the same as $\cos\theta$

$\vec{a}\cdot\vec{b} < 0$, θ is obtuse



The dot product measures the extent to which $\vec{a}$ and $\vec{b}$ point in the same direction.

Directional angles



$\vec{a} = \langle a_1, a_2, a_3 \rangle$

$\cos\gamma$, $\cos\alpha$ and $\cos(\beta)$ are called directional cosines.

$$\cos\alpha = \frac{\vec{a} \cdot \langle 1,0,0 \rangle}{|a|\,|i|} = \frac{a_1}{|a|}$$

$$\cos\beta = \frac{a_2}{|a|} \quad \text{and} \quad \cos\gamma = \frac{a_3}{|a|}$$

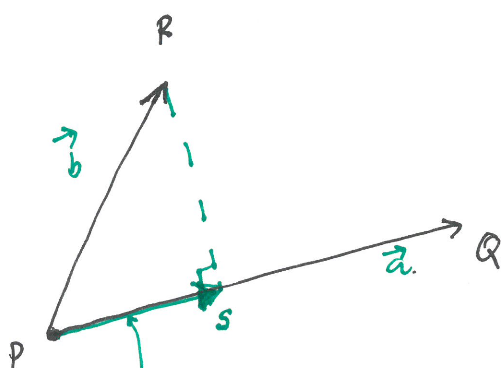
$$\vec{a} = \langle a_1, a_2, a_3 \rangle = \langle |a|\cos\alpha,\ |a|\cos\beta,\ |a|\cos\gamma \rangle$$
$$= |a| \langle \cos\alpha,\ \cos\beta,\ \cos\gamma \rangle$$

$$\Rightarrow \frac{\vec{a}}{|a|} = \langle \cos\alpha,\ \cos\beta,\ \cos\gamma \rangle$$

The directional cosines are the components of the unit vectors.

Projections

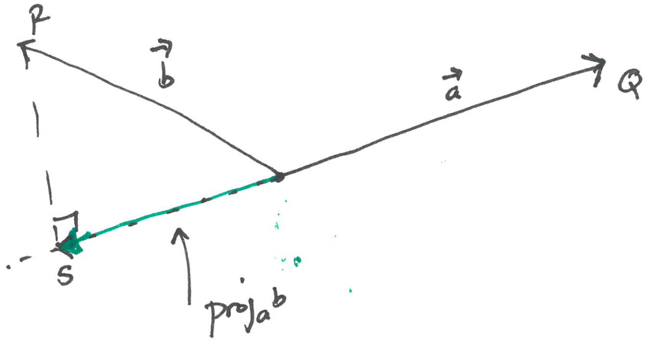
Visually - this is the shadow of b along PQ



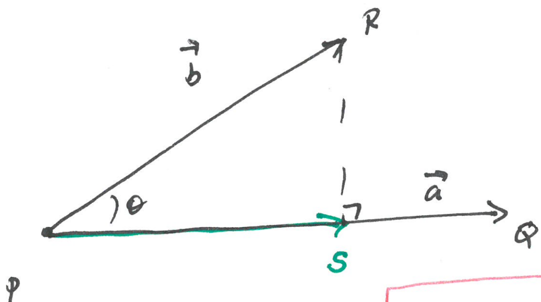
given
- $\vec{a} = \overrightarrow{PQ}$
- $\vec{b} = \overrightarrow{PR}$
- If we drop a perpendicular from R to the line containing $\overrightarrow{PQ}$, to S.

$\overrightarrow{PS} = \text{proj}_{\vec{a}}\vec{b}$.

$\text{proj}_a b$ - projection of of b onto a.



Scalar projection / Component of $\vec{b}$ along $\vec{a}$.



$$\cos\theta = \frac{|\vec{PS}|}{|\vec{b}|} \Rightarrow \boxed{|\vec{b}|\cos\theta = |\vec{PS}|} = \text{magnitude of projection vector.}$$

Recall the dot product

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta = |\vec{a}|\left(|\vec{b}|\cos\theta\right) \Rightarrow \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = |b|\cos\theta$$

$$\Rightarrow |\vec{PS}| = |\vec{b}|\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = \frac{\vec{a}}{|\vec{a}|}\cdot\vec{b}$$

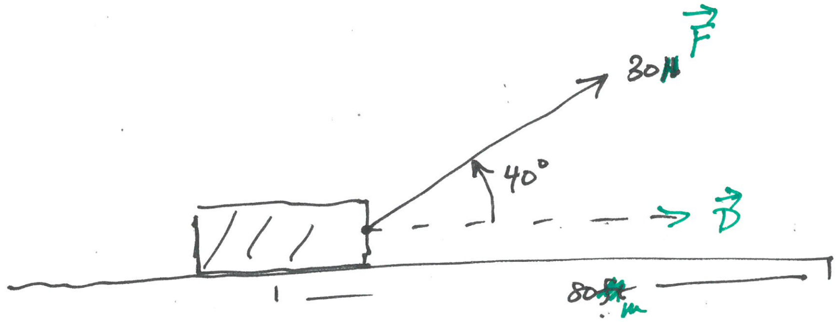
The scalar projection of $\vec{b}$ onto $\vec{a}$ $= \frac{\vec{a}}{|\vec{a}|}\cdot\vec{b}$

unit vectors

Vector projection of $\vec{b}$ onto $\vec{a}$ $= \text{proj}_a \vec{b} = \underbrace{\left(\frac{\vec{a}\cdot\vec{b}}{|a|}\right)}_{\text{Scalar projection}} \frac{\vec{a}}{|\vec{a}|} = \frac{\vec{a}\cdot\vec{b}}{|a|^2}\,\vec{a}.$

A sled is pulled along a level path through snow by a rope.

A 30-lb force acting at an angle of 40° above the horizontal moves the sled 80ft. Find the work done by the force.



Find the work done by the force.

Work done

Scalar component in direction of displacement $= |\vec{F}| \cos(40°)$.

Work done $= |F| \cos(40°) |\vec{D}| = \underline{30 \cos(40°) \cdot 80}$ Nm.

$= \vec{F} \cdot \vec{D}$.

Examples

Given $\vec{a} = 4i - 3j + k$ and $\vec{b} = 2i - k$.

(i) Find the angle between $\vec{a}$ and $\vec{b}$.

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{\langle 4,-3,1\rangle\cdot\langle 2,0,-1\rangle}{|\langle 4,-3,1\rangle|\cdot|\langle 2,0,-1\rangle}$$

$$|\langle 4,-3,1\rangle| = \sqrt{4^2+(-3)^2+1^2} = \sqrt{16+9+1} = \sqrt{26}$$

$$|\langle 2,0,-1\rangle| = \sqrt{2^2+0^2+(-1)^2} = \sqrt{5}$$

$$\cos\theta = \frac{8+0-1}{\sqrt{26}\cdot\sqrt{5}} = \frac{7}{\sqrt{130}} \Rightarrow \theta = \cos^{-1}\left(\frac{7}{\sqrt{130}}\right). \simeq 52°.$$

(ii) Find the scalar and vector projection of $\vec{b}$ onto $\vec{a}$.

$$\text{Scalar projection} = \frac{\vec{a}\cdot\vec{b}}{|a|} = \frac{\langle 4,-3,1\rangle\cdot\langle 2,0,-1\rangle}{\sqrt{26}} = \frac{7}{\sqrt{26}}.$$

$$\text{Vector projection} = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\right)\frac{\vec{a}}{|a|} = \frac{7}{\sqrt{26}}\cdot\frac{1}{\sqrt{26}}\langle 4,-3,1\rangle$$

$$= \frac{7}{26}\langle 4,-3,1\rangle$$

~~Rk~~ Scalar projection of $\vec{b}$ along $\vec{a}$ is also known as ~~ten~~

$$\text{comp}_{a}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$$