12.4 The Cross - Product

Given $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$ we want to find $\vec{c}$ (non-zero) that is perpendicular to both $\vec{a}$ and $\vec{b}$.

Let $\vec{c} = \langle c_1, c_2, c_3 \rangle$ be perpendicular to both $\vec{a}$ and $\vec{b}$, then

$$\vec{a}\cdot\vec{c} = 0 \quad \text{and} \quad \vec{b}\cdot\vec{c} = 0.$$

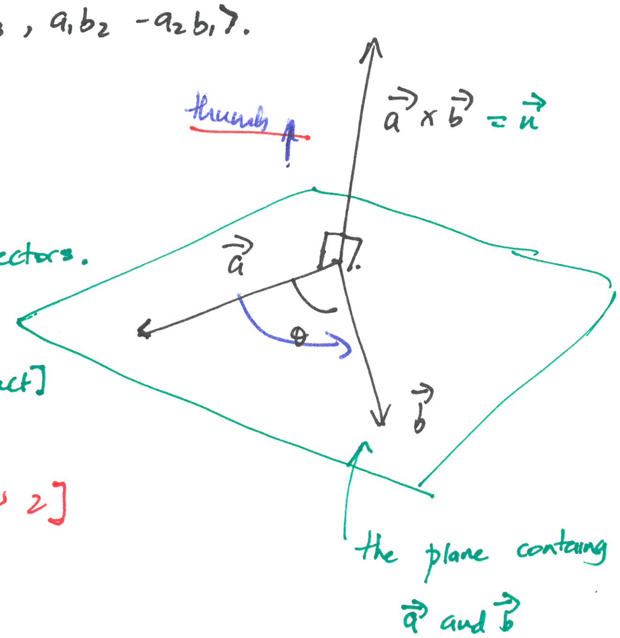
$$\boxed{\begin{aligned} \vec{a}\cdot\vec{c} &= a_1c_1 + a_2c_2 + a_3c_3 = 0 \\ \vec{b}\cdot\vec{c} &= b_1c_1 + b_2c_2 + b_3c_3 = 0 \end{aligned}}$$

Solve these 2 equations.

Definition

If $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$, then the cross product of $\vec{a}$ and $\vec{b}$, $\vec{a}\times\vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$.

$$\vec{a}\times\vec{b} = \langle a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1 \rangle.$$



Remarks

1. The cross product, $\vec{a}\times\vec{b}$ is a vector
2. We can only define $\vec{a}\times\vec{b}$ for 3 dimensional vectors.

Definition (Determinant) [calculation of vector product]

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

[Determinant of order 2]

Determinant of order 3.

$$\begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \end{array} = a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} i - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} j + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} k$$

This comes from noticing that

$$\vec{a} \times \vec{b} = \langle a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1 \rangle.$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \quad - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \quad \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}.$$

Example

Let $\vec{a} = \langle 6, 0, -2 \rangle$ and $\vec{b} = \langle 0, 8, 0 \rangle$.

$$\begin{array}{ccc} + & - & + \end{array}$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ 6 & 0 & -2 \\ 0 & 8 & 0 \end{vmatrix}$$

$$= i \begin{vmatrix} 0 & -2 \\ 8 & 0 \end{vmatrix} - \begin{vmatrix} 6 & -2 \\ 0 & 0 \end{vmatrix} j + \begin{vmatrix} 6 & 0 \\ 0 & 8 \end{vmatrix} k$$

$$= (0 - (-16)) i - (0 - (-2)) j + (72 - 0) k$$

$16i + 2j + 72k$, [$\langle 16, 2, 72 \rangle$ is orthogonal to both $\langle 6, 0, -2 \rangle$ and $\langle 0, 8, 0 \rangle$]

FACT (length of $\vec{a} \times \vec{b}$)

If θ is the angle between $\vec{a}$ and $\vec{b}$ $(0 \le \theta \le \pi)$ then

$$|\vec{a} \times \vec{b}| = |\vec{a}| |\vec{b}| \sin\theta$$

Summary

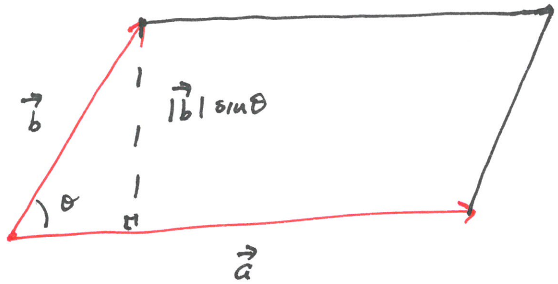
$\vec{a} \times \vec{b}$ is the vector perpendicular to both $\vec{a}$ and $\vec{b}$, orientation is determined by the right hand rule, and the length is $|\vec{a}| |\vec{b}| \sin\theta$.

FACT

Two vectors $\vec{a}$ and $\vec{b}$ are parallel if and only if

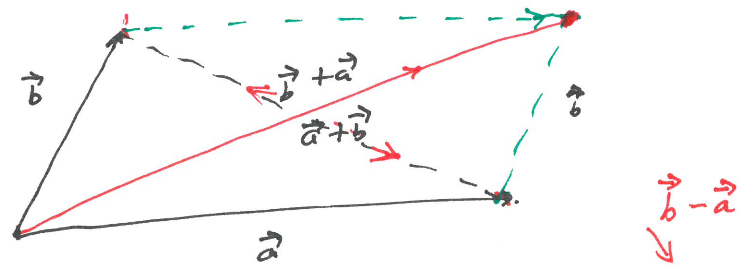
$$\vec{a} \times \vec{b} = 0.$$

$$|\vec{a} \times \vec{b}| = |\vec{a}| |\vec{b}| \sin(\pi) = 0.$$

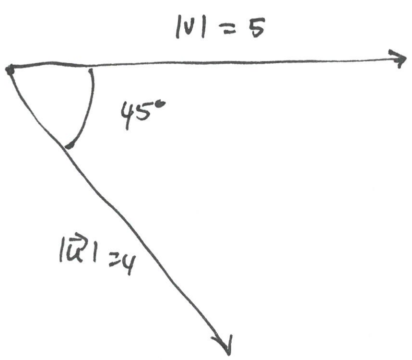


$$A = |\vec{a}|(|\vec{b}| \sin\theta) = |\vec{a} \times \vec{b}|$$

The area length of the cross-product $|\vec{a} \times \vec{b}|$ is equal to the area of the parallelogram determined by $\vec{a}$ and $\vec{b}$



Example #1



$$|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin(45°).$$

points out of the plane.

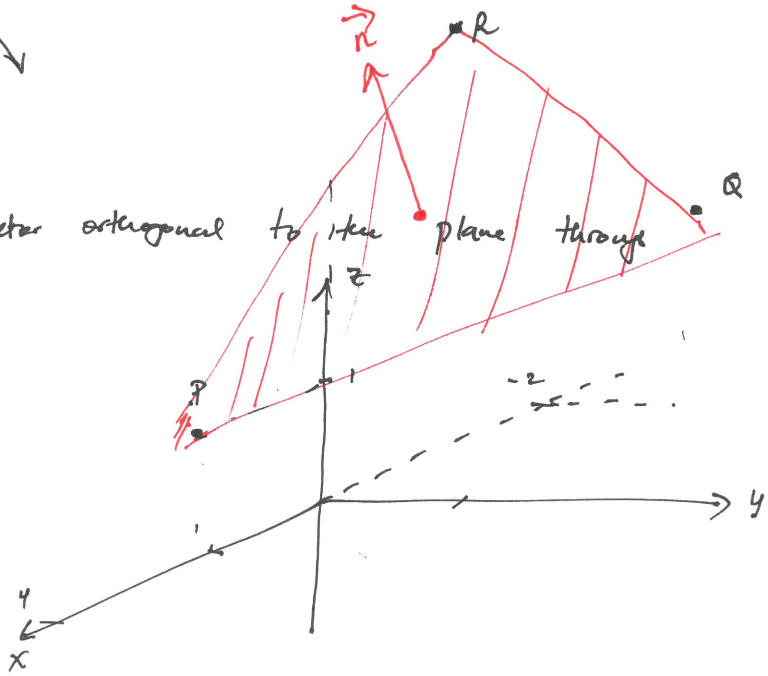
Example #2

12.4 #29. Find the nonzero vector orthogonal to the plane through

P (1,0,1)

Q (−2,1,3)

R (4,2,5)



Let $\vec{u} = \vec{PQ} = \langle -2-1,\ 1-0,\ 3-1 \rangle = \langle -3, 1, 2 \rangle$

$\vec{v} = \vec{PR} = \langle 4-1,\ 2-0,\ 5-1 \rangle = \langle 3, 2, 4 \rangle$.

$$\boxed{\vec{n} = \vec{u} \times \vec{v}}$$

$$\begin{matrix} i & j & k \\ -3 & 1 & 2 \\ 3 & 2 & 4 \end{matrix}$$

$$i(4-4) - j(-12-6) + k(-6-3)$$

$18j - 9k = \langle 0, 18, -9 \rangle$ is normal to the plane containg PQR. infact any scalar multiple of $\langle 0, 18, -9 \rangle$ is perpendicular to the plane.

Properties of the cross-product

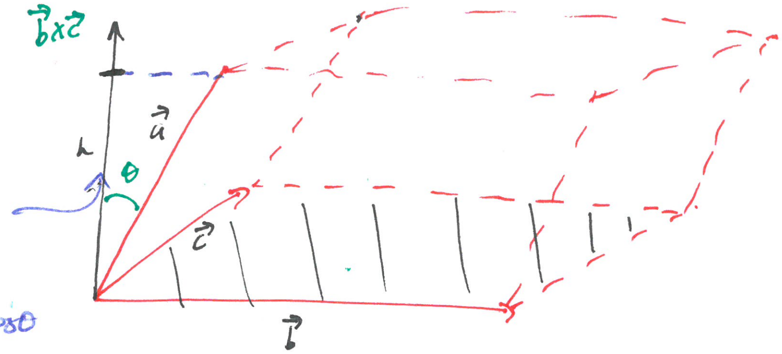
If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are vectors and α is a scalar, then

1. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
2. $(\alpha\vec{a}) \times \vec{b} = \alpha(\vec{a} \times \vec{b}) = \vec{a} \times (\alpha\vec{b})$
3. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$.
4. $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$
5. $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$
6. $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

Ignore.

Triple products

$\vec{a}\cdot(\vec{b}\times\vec{c})$



perpendicular height is $|\vec{a}|\cos\theta$

~~$\vec{a}\cdot(\vec{b}\times\vec{c}) = a_1(b_2$~~

Area of base $=$ $|\vec{b}\times\vec{c}|$

Volume of parallelepiped $=$ Area of base $\times$ perpendicular height.

$h = |a|\,|\cos\theta|$.

$V = |\vec{b}\times\vec{c}|\,|a|\cos\theta = |(\vec{b}\times\vec{c})\cdot\vec{a}|$

Recall the dot product

$\boxed{\vec{u}\cdot\vec{v} = |\vec{u}|\,|\vec{v}|\cos\theta.}$

Application

Show that $\vec{u} = i+5j-2k$, $\vec{v} = 3i-j$ and $\vec{w} = 5i+9j-4k$ are coplanar.

We will compute $\vec{u}\cdot(\vec{v}\times\vec{w})$

$$(\vec{v}\times\vec{w}) = \begin{vmatrix} i & j & k \\ 3 & -1 & 0 \\ 5 & 9 & -4. \end{vmatrix} = i(4-0) - j(-12-0) + k(27+5)$$

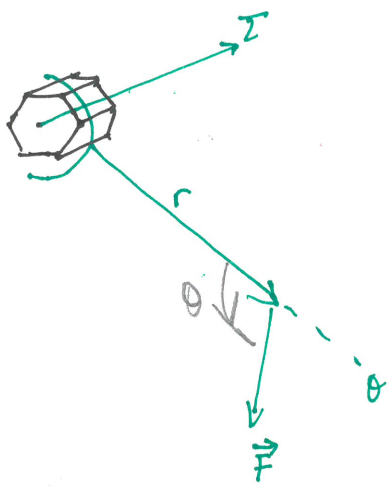
$$= 4i + 12j + 32k.$$

$$\vec{u}\cdot(\vec{v}\times\vec{w}) = \langle 1, 5, -2\rangle \cdot \langle 4, 12, 32\rangle$$

$$= 4 + 60 - 64 = 0.$$

This means that $\vec{u}$, $\vec{v}$ and $\vec{w}$ are coplanar.

Torque



Torque τ is the ~~x~~ dot product of the position vector r and Force

$$|\tau| = |r \times F| = |r|\,|F| \sin\theta.$$