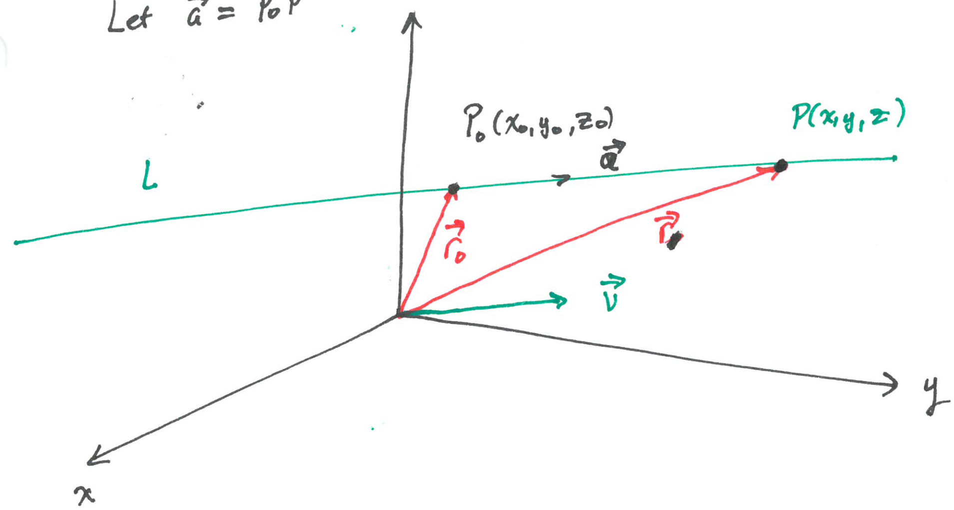
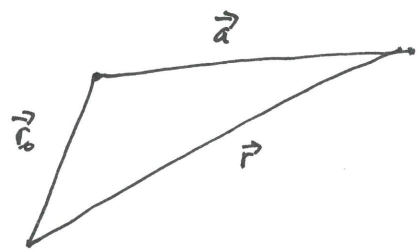
12.5 **Equations of lines and planes**

Suppose we know a point $P_0(x_0, y_0, z_0)$ on L and the direction of L.

Let $\vec{a} = \overrightarrow{P_0P}$



$\vec{r_0}$ and $\vec{r}$ are position vectors of $P_0(x_0, y_0, z_0)$ and $P(x, y, z)$ respectively.



$r = \vec{r_0} + \vec{a}$

Recall that ~~$\vec{a} = t\vec{n}$~~ $\vec{a} \parallel \vec{v}$, therefore $\vec{a} = t\vec{v}$.

$\boxed{r = r_0 + t\vec{v}}$

- This is the vector equation of L.
- Parameterized by t.

$t = 0$ represents the point $P_0(x_0, y_0, z_0)$

$t < 0$ represents the points before P_0

$t > 0$ represents the points after P_0.

In component form:

$$\langle x, y, z\rangle = \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle$$

$$\left.\begin{aligned} x &= x_0 + at \quad ① \\ y &= y_0 + bt \quad ② \\ z &= z_0 + ct \quad ③ \end{aligned}\right\}$$

$t \in \mathbb{R}$

- These are the parametric equations of the line L.

12.5 **Symmetric Equations**

If we ~~eff~~ eliminate the parameter t

$t = \dfrac{x - x_0}{a}$, $t = \dfrac{y - y_0}{b}$, $t = \dfrac{z - z_0}{c}$, we get

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Remark

The line passes through 2 points with position vectors $\vec{r_0}$ and $\vec{r_1}$.

Take $\vec{v} = \vec{r_1} - \vec{r_0}$ so that

$$\boxed{\begin{aligned}\vec{r} &= \vec{r_0} + t(\vec{r_1} - \vec{r_0}) \\ &= (1-t)\vec{r_0} + t\vec{r_1}\end{aligned}} \qquad 0 \le t \le 1.$$

Example

1. Find the line equation of the line through the point $(0, 14, -10)$ and parallel to $x = -1+2t$, $y = 6-3t$, $z = 3+9t$.

$$\vec{r} = \vec{r_0} + t\vec{v}$$

~~let~~ $\vec{v} = \langle 2, -3, 9 \rangle$. — direction of our line

$\vec{r_0} = \langle 0, 14, -10 \rangle$

therefore

$$\boxed{r(t) = \langle 0, 14, -10 \rangle + t\langle 2, -3, 9 \rangle.}$$

In parametric form

$x = 0 + 2t$

$y = 14 - 3t$

$z = -10 + 9t$

2. **Remark** point r_0 and direction $\vec{v}$

Symmetric Equation

$$\frac{x}{2} = \frac{14 - y}{3} = \frac{z + 10}{9}$$

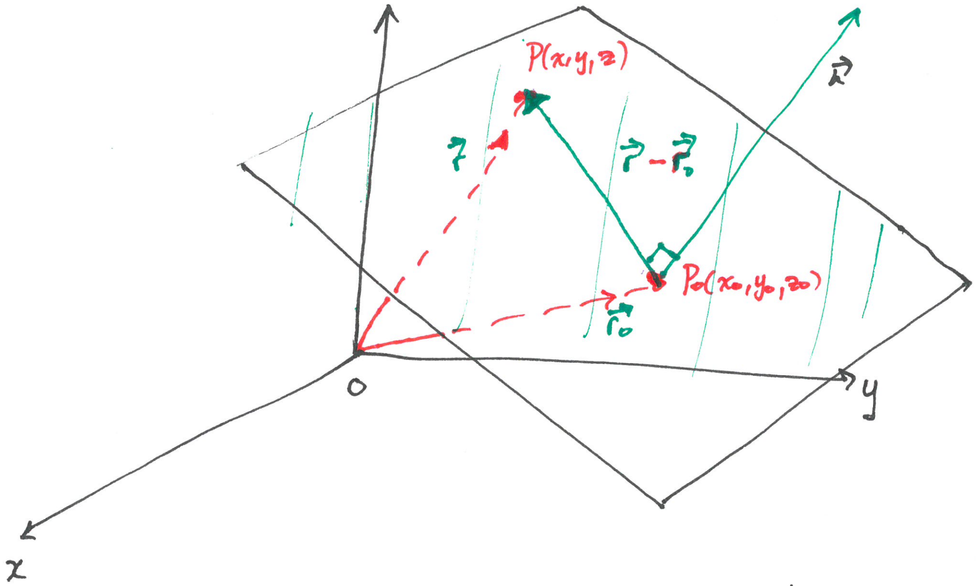
Planes 12.5

Lines are given by a position vector $\vec{r}_0$ and a direction $\vec{v}$.

$$r(t) = \vec{r}_0 + t\vec{v}.$$

We will determine planes by a point $P_0(x_0, y_0, z_0)$ and $\vec{n}$

Let $P(x,y,z)$ be an arbitrary point on the plane.



$\vec{r}_0 = \overrightarrow{OP_0}$

$\vec{r} = \overrightarrow{OP}$

$\vec{r} - \vec{r}_0 = \overrightarrow{P_0P}.$

- The normal vector $\vec{n}$ is orthogonal to every vector in the given plane.

For any point P_0, $n \perp \vec{r} - \vec{r}_0$

$$\boxed{n \cdot (\vec{r} - \vec{r}_0) = 0. \quad \Rightarrow \quad \vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0}$$ vector equation

This is the vector equation of a plane.

In component form:

$\vec{n} = \langle a, b, c \rangle$ $\quad \vec{r} = \langle x, y, z \rangle$, $\quad \vec{r}_0 = \langle x_0, y_0, z_0 \rangle$

$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$ gives

$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$ [scalar equation]

Expanding

$$ax - ax_0 + by - by_0 + cz - cz_0 = 0$$

$$ax + by + cz + (-ax_0 - by_0 - cz_0) = 0$$ [linear equation]

Def'n Linear equation of a plane $\left[\, ax + by + cz + d = 0 \,\right]$, $d = -ax_0 - by_0 - cz_0$

Example #1

Find the plane passing through $(5,3,5)$ with normal vector $2i+j-k$

$\vec{n} = \langle 2, 1, -1 \rangle$ $\quad\quad$ $\vec{r}-\vec{r_0} = \langle x-5, y-3, z-5 \rangle$

$\vec{n} \cdot \vec{r}-\vec{r_0} = 0 \quad \Rightarrow \quad \langle 2,1,-1 \rangle \cdot \langle x-5, y-3, z-5 \rangle = 0$

$$2(x-5) + 1(y-3) - 1(z-5) = 0$$

$$2x - 10 + y - 3 - z + 5 = 0$$

$$2x + y - z - 8 = 0$$

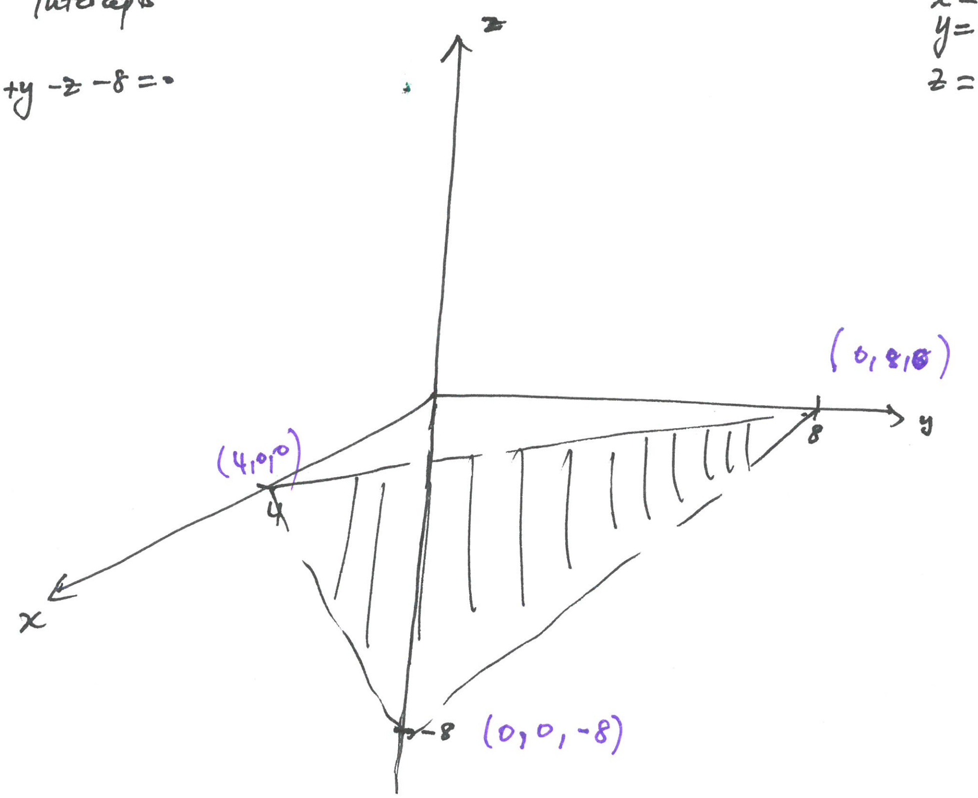
Notice that in the linear equation of the plane, the coefficients of x, y and z correspond to the components of the normal vector $\vec{n} = \langle 2, 1, -1 \rangle$.

Sketch of plane

Find intercepts

$2x + y - z - 8 = 0$

$x=0$	$x=0$	$y=0$
$y=0$	$z=0$	$z=0$
$z=-8$	$y=8$	$x=4$



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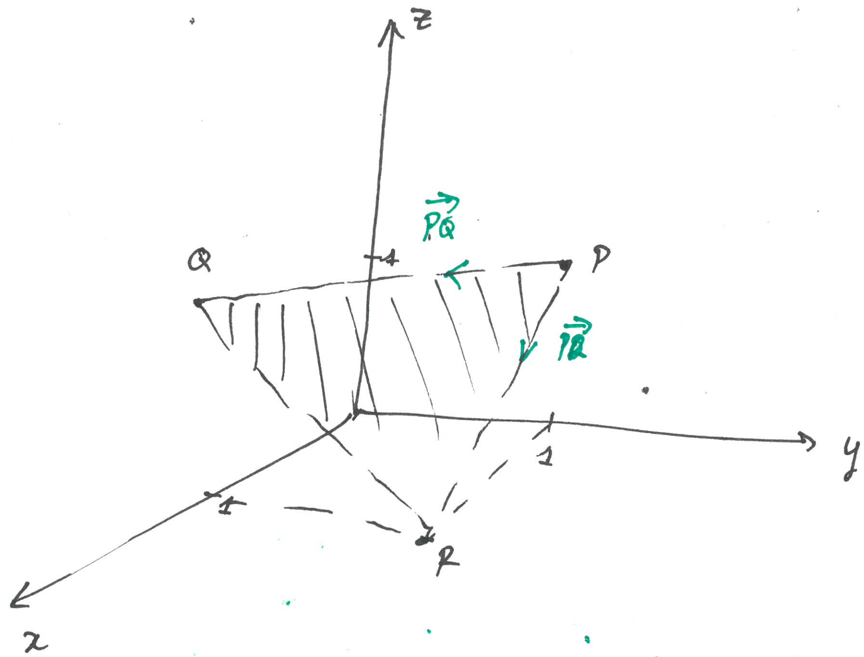
#31 Find the equation of the plane that passes through $\underset{P}{(0,1,1)}$, $\underset{Q}{(1,0,1)}$ and $\underset{R}{(1,1,0)}$ #5



$\vec{PQ} = \langle 1-0,\ 0-1,\ 1-1 \rangle = \langle 1,-1,0 \rangle$

$\vec{PR} = \langle 1-0,\ 1-1,\ 0-1 \rangle = \langle 1,0,-1 \rangle$

The normal vector $\vec{n}$ is perpendicular to every vector in the plane so $\vec{n}$ is orthogonal to $\vec{PQ}$ and $\vec{PR}$

$\vec{n} = \vec{PQ} \times \vec{PR}$.

$$\vec{n} = \begin{vmatrix} i & j & k \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = i(1-0) - j(-1) + k\left(\overset{1-0}{\cancel{0+1}}\right)$$

$i + j + k$

$\vec{n} = \langle 1, 1, +1 \rangle$

Equation of plane:

$\vec{n} \cdot (\vec{r} - \vec{r_0})$

$\vec{r_0}$ is an the position vector of any point on the plane, pick $P = (0,1,1)$

$\vec{r_0} = \langle 0,1,1 \rangle$

$\vec{n} \cdot (\vec{r} - \vec{r_0}) = 0 \Rightarrow \langle 1,1,+1 \rangle \cdot \langle x-0,\ y-1,\ z-1 \rangle = 0$

$\langle 1,1,-1 \rangle \cdot \langle x,\ y-1,\ z-1 \rangle = 0 \Rightarrow x + (y-1) + (z-1) = 0.$ / $x+y+z=2.$

12.5

45 Find the point at which the line

$$x = 3-t, \; y = 2+t, \; z = 5t$$

intersects the plane

$$x - y + 2z = 9.$$

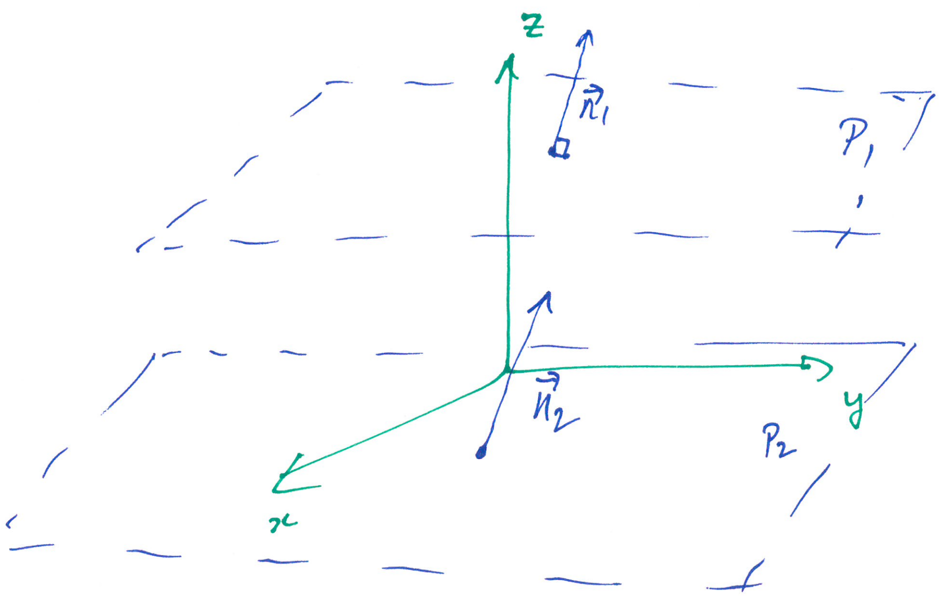
At the point of intersection, t

$$(3-t) - (2+t) + 2(5t) = 9$$
$$3 - t - 2 - t + 10t = 9$$
$$1 + 8t = 9$$
$$8t = 8 \Rightarrow t = 1.$$

The plane and the line intersect at $t=1 \Rightarrow$
$$x = (3-1) = 2$$
$$y = (2+1) = 3$$
$$z = 5(1) = 5$$
$$\boxed{(2,3,5)}$$

Parallel Planes

- 2 planes are parallel if their normal vectors are parallel.



$\vec{n}_1 = k\vec{n}_2$ if

P_1 and P_2 are parallel planes.

(51) 12.5

Eg Determine whether the planes are parallel, perpendicular or neither. #7

$P_1 \Rightarrow x+4y-3z=1$

$P_2 \Rightarrow -3x+6y+7z=0$

The normal vector for P_1 is. $\vec{n_1} = \langle 1,4,-3\rangle$

P_2 $\vec{n_2} = \langle -3,6,7\rangle$.

Just compute $\vec{n_1}\cdot\vec{n_2}$ to see if the vectors are perpendicular.

Let θ be the angle between the planes

$$\cos\theta = \frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}||\vec{n_2}|} = \frac{\langle 1,4,-3\rangle\cdot\langle -3,6,7\rangle}{|\langle 1,4,-3\rangle|\,|\langle -3,6,7\rangle|}$$

$$= \frac{-3+24-21}{\sqrt{26}\cdot\sqrt{94}} = \frac{0}{\sqrt{26}\cdot\sqrt{94}}$$

$|\langle 1,4,-3\rangle|$

$= \sqrt{1^2+4^2+(-3)^2} = \sqrt{1+16+9} = \sqrt{26}$

$|\langle -3,6,7\rangle| = \sqrt{(-3)^2+6^2+7^2} = \sqrt{94}$

P_1 and P_2 are perpendicular!

No need for this

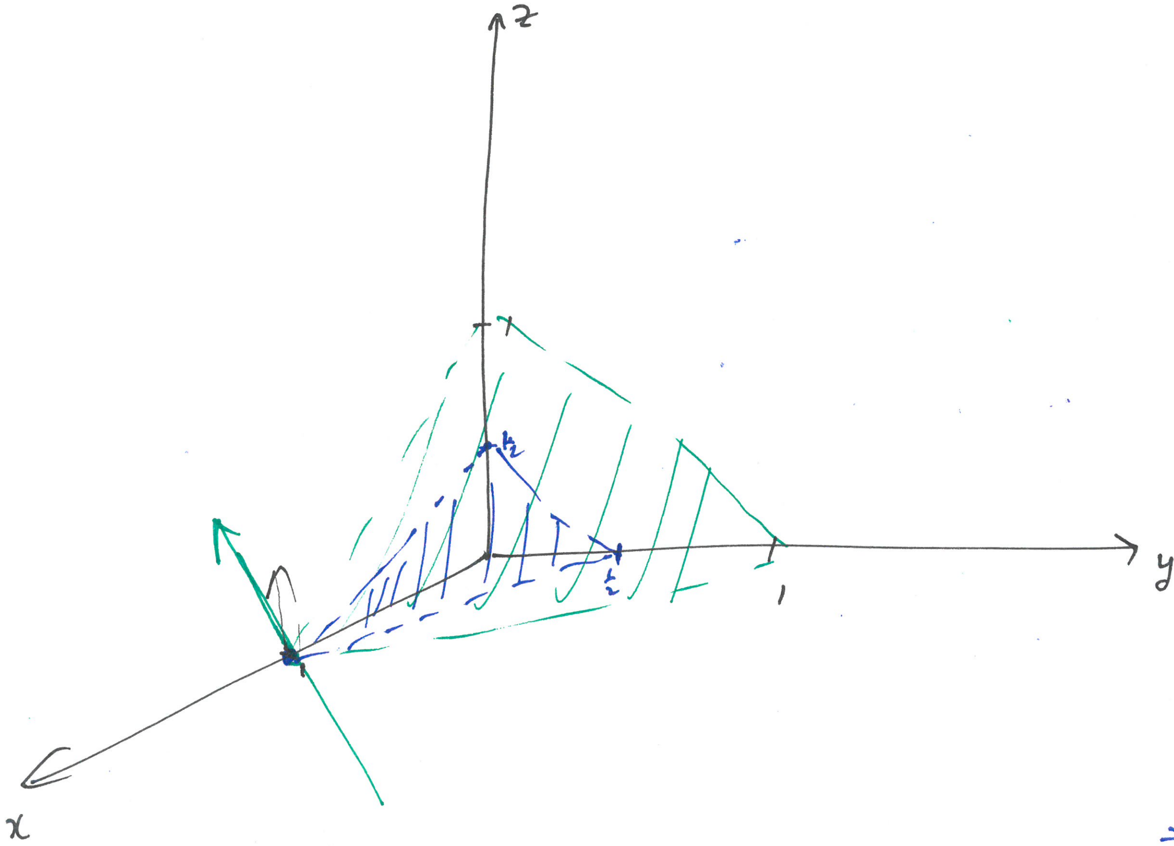
If $\vec{n_1}\cdot\vec{n_2} \neq 0$, then we need $|\vec{n_1}|$, $|\vec{n_2}|$ to compute the angle

12.5

#8

(57) Find the parametric equations for the line of intersection of

$x+y+z=1$ and $x+2y+2z=1$.



We need to find a point on the line and a direction vector $\vec{v}$.

Both planes pass through the x-y plane $(z=0)$

$x+y=1$ ① and $x+2y=1$. ②

Now, we have 2 equations, and 2 unknowns

From ①: $y=1-x$. plug into ② $x+2(1-x)=1$

$x+2-2x=1$

$-x+2=1 \Rightarrow \underline{x=1}$

from ① $y=1-x \Rightarrow y=0. \Rightarrow$ a point on the line is $(1,0,0)$

Now for the direction $\vec{v}$

Since L lies on both planes, $\vec{v}$ is orthogonal to $\vec{n_1}$ and $\vec{n_2}$

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#9

So $\vec{v} = \vec{n_1} \times \vec{n_2}$

$$\vec{v} = \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{ccc} + & - & + \\ \end{array}$$

$$\vec{v} = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{vmatrix} = i\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} - j\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} + k\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}$$

$\vec{v} = 0i - j + k$, $\vec{v} = \langle 0, -1, 1 \rangle$.

$\vec{r}(t) = \vec{r_0} + t\vec{v}$

$\langle 1, 0, 0 \rangle + t\langle 0, -1, 1 \rangle$

$$\boxed{\begin{aligned} x &= 1 + t\cdot 0 \\ y &= 0 - t \\ z &= 0 + t \end{aligned}}$$

Parametric Equation

~~Symmetric equation~~

Exercise . find the angle between the planes!

$\cos\theta = \dfrac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}||\vec{n_2}|} = \dfrac{1}{14}$, $\theta = \cos^{-1}\left(\frac{1}{14}\right) \approx 88.°$ $89.6°$

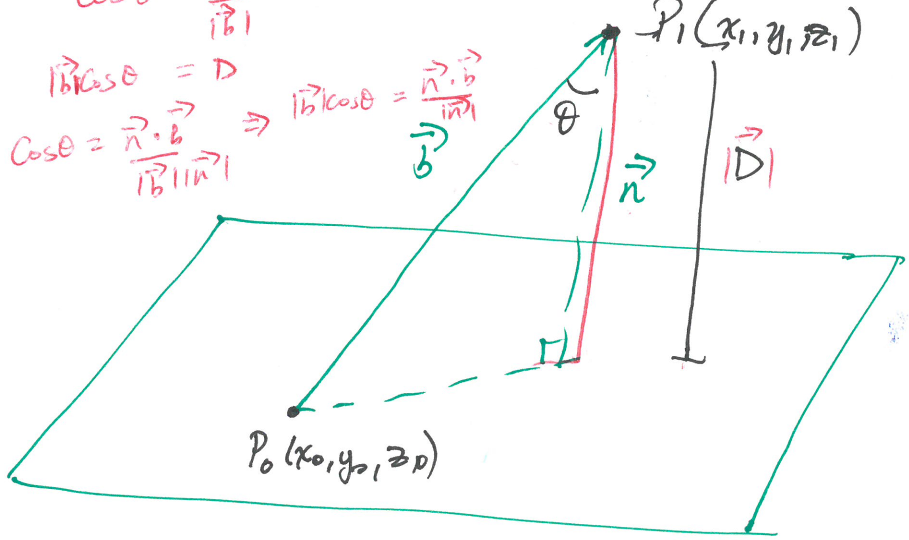
The distance between a point $P_1(x_1, y_1, z_1)$ and a plane

$ax + by + cz + d = 0$

$\cos\theta = \dfrac{D}{|\vec{b}|}$

$|\vec{b}|\cos\theta = D$

$\cos\theta = \dfrac{\vec{n}\cdot\vec{b}}{|\vec{b}||\vec{n}|} \Rightarrow |\vec{b}|\cos\theta = \dfrac{\vec{n}\cdot\vec{b}}{|\vec{n}|}$



$\vec{n} = \langle a, b, c \rangle$

$\vec{b} = \overrightarrow{P_0P_1}$

$= \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$

[what about the distance between 2 parallel planes?].

Find a point on second plane & calculate distance

$|\vec{D}|$ is the size of the projection of $\vec{b}$ onto $\vec{n}$

$$|\vec{D}| = \text{comp}_{n}\vec{b} = \frac{|\vec{n}\cdot\vec{b}|}{|\vec{n}|} = \frac{a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)}{\sqrt{a^2 + b^2 + c^2}}$$

12.8

Show that the lines L_1 and L_2 with parametric equations #10

$x = 1+t \quad y = -2+3t \quad z = 4-t$

$x = 2s \quad y = 3+s \quad z = -3+4s$

are skew; i.e, they do not intersect, and are not parallel.

Worksheet

$\underline{L_1}$ - $\vec{V_1} = \langle 1, 3, -1 \rangle$

L_2 - $\vec{V_2} = \langle 2, 1, 4 \rangle$

$\vec{V_1}$ is not parallel to $\vec{V_2}$, therefore L_1 is not parallel to L_2.

Suppose the lines intersect

$1+t = 2s \quad \dots \quad (i)$

$-2+3t = 3+s \quad \dots \quad (ii)$

from equation (i) $t = 2s-1$, plug into (ii)

$-2+3(2s-1) = 3+s$

$-2+6s-3 = 3+s$

$6s-5 = 3+s$

$5s = 8$

$s = \frac{8}{5}$

$t = 2 \cdot \frac{8}{5} - 1 = \frac{16}{5} - 1 = \frac{11}{5}$

Check the z-coordinate

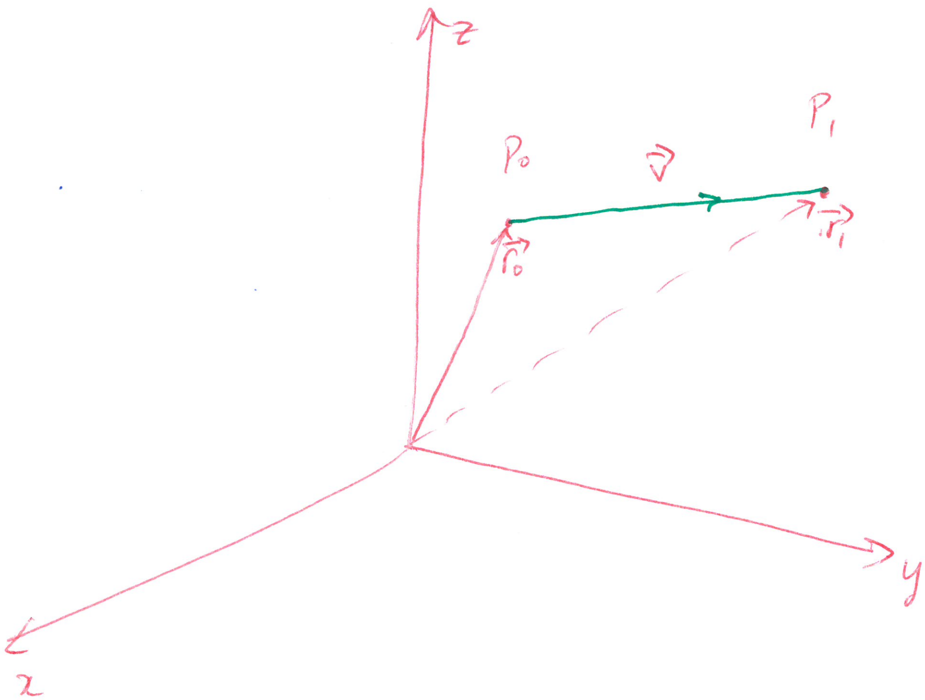
$z = 4 - \frac{11}{5}$ for L_1 $= 1.8$

$z = -3 + 4 \cdot \frac{8}{5}$ for L_2.

$-3 + \frac{32}{5} = 5$

Since the z-coordinate are $\underline{\text{not}}$ equal. The lines do not $\underline{\text{intersect}}$

Parameterization of a line segment.



$\vec{r}(t) = \vec{r_0} + t\vec{v}$

$v = \vec{r_1} - \vec{r_0}$

$\vec{r}(t) = \vec{r_0} + t(\vec{r_1} - \vec{r_0})$

$\vec{r}(t) = \vec{r_0} + t\vec{r_1} - t\vec{r_0}$

$\vec{r}(t) = (1-t)\vec{r_0} + t\vec{r_1} \quad , \quad 0 \leq t \leq 1$

Check: when $t=0$

$\vec{r}(t) = \vec{r_0}$

when $t=1$

$\vec{r}(t) = \vec{r_1}$!