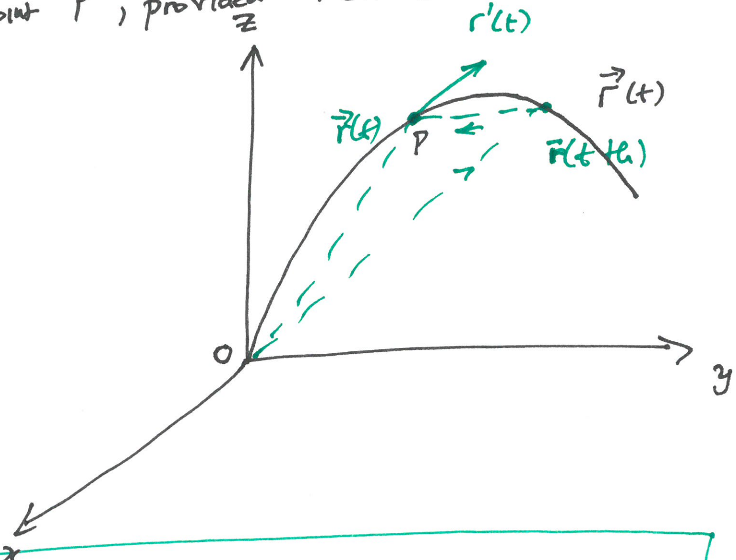
Derivatives of vector functions

Given $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$.

$$\frac{d\vec{r}}{dt} = r'(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$, provided the limits exist.

$r'(t)$ is the tangent vector to the curve defined by $\vec{r}(t)$ at the point P, provided $r'(t)$ exists and $r'(t) \neq 0$.



Unit tangent vector $T(t) = \dfrac{\vec{r}'(t)}{|r'(t)|}$

FACT

If $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, then $\vec{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$

Rules

1. $\dfrac{d}{dt}\left[\vec{u}(t) + \vec{v}(t)\right] = \vec{u}'(t) + \vec{v}'(t)$
2. $\dfrac{d}{dt}\left[c\,\vec{u}(t)\right] = c\,u'(t)$
3. $\dfrac{d}{dt}\left[f(t)\,\vec{u}(t)\right] = f'(t)\,\vec{u}(t) + f(t)\,\vec{u}'(t)$ [f is a scalar function]
4. $\dfrac{d}{dt}\left[\vec{u}(t) \cdot \vec{v}(t)\right] = \vec{u}'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}'(t)$
5. $\dfrac{d}{dt}\left[\vec{u}(t) \times \vec{v}(t)\right] = u'(t) \times \vec{v}(t) + \vec{u}(t) \times v'(t)$

6 $\frac{d}{dt}[\vec{u}(f(t))] = f'(t)\,\vec{u}'(f(t))$.

Example #1.

Find $\vec{r}'(t)$ for $\vec{r}(t) = \langle t-2,\ t^2+1\rangle$ at $t=-1$

$\vec{r}'(t) = \langle 1,\ 2t\rangle$

$r'(-1) = \langle 1, -2\rangle$.

Unit tangent vector

$|\langle 1, -2\rangle| = \sqrt{1^2+(-2)^2} = \sqrt{5}$.

Unit tangent vector $= \frac{1}{\sqrt{5}}\langle 1, -2\rangle$.

Example #2

$\vec{r}(t) = \cos t\, i + 3t\, j + 2\sin(2t)\, k$ at $t=0$

$\vec{r}'(t) = -\sin(t)\, i + 3j + 4\cos(2t)k$

$r'(0) = -\sin(0)\, i + 3j + 4\cos(0)k$

$= 0i + 3j + 4k$

$\vec{T}$, the unit tangent vector $= \frac{r'(0)}{|\vec{r}'(0)|}$

$|r'(0)| = |\langle 0, 3, 4\rangle|$

$= \sqrt{3^2+4^2} = \sqrt{9+16} = \sqrt{25} = 5$

$T = \frac{1}{5}\langle 0, 3, 4\rangle$.

13.2

$$\boxed{\text{The second derivative } \vec{r}\,''(t) = (\vec{r}\,'(t))'}$$

Integrals

Let $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

$$\int_a^b \vec{r}(t)\,dt = \left(\int_a^b f(t)\,dt\right) i + \left(\int_a^b g(t)\,dt\right) j + \left(\int_a^b h(t)\,dt\right) k$$

Example

$$\int (ti - t^3 j + 3t^5 k)\,dt = \left(\int t\,dt\right) i - \left(\int t^3\,dt\right) j + \left(\int 3t^5\,dt\right) k$$

$$= \left(\frac{t^2}{2}\right) i - \left(\frac{t^4}{4}\right) j + \left(\frac{3t^6}{6}\right) k + \vec{C}$$

$\vec{C}$ is a constant vector [independent of $\vec{t}$]

13.2 #4

(24) Find the parametric equations for the tangent line to the curve with para the given parametric equations.

$$x = e^t, \quad y = te^t, \quad z = te^{t^2} \quad \text{at} \quad (1,0,0)$$

To find the equation of a line → point $(1,0,0)$ vector $\vec{v}$

$$\vec{v} = \vec{r}\,'(t) = \left\langle \frac{d}{dt}(e^t), \frac{d}{dt}(te^t), \frac{d}{dt}(te^{t^2}) \right\rangle \Big|_{@\, t \text{ correspondin to } \langle 1,0,0\rangle}$$

$$= \langle e^t, (te^t + e^t, t \cdot e^{t^2} \cdot 2t + e^{t^2} \rangle$$

$$= \langle e^t, te^t + e^t, 2t^2 e^{t^2} + e^{t^2} \rangle$$

We need to find t @ $(1,0,0)$

$e^t = 1 \Rightarrow t = \ln(1)$. [you can confirm that $t = \ln(1)$ satisfies the rest of the points].

$$\vec{v}(t) = r'(\ln(1)) = \langle e^{\ln(1)}, 0 + e^{\ln(1)}, 0 + e^{(\ln(1))^2} \rangle$$

$$= \langle 1, 1, 1 \rangle$$

The tangent line has equation

$$\vec{r}_T(t) = \langle 1,0,0 \rangle + t\langle 1,1,1 \rangle$$

$$x = 1 + t$$
$$y = t$$
$$z = t.$$

(34) 13.2 #5

At what point do the curves

$$\vec{r}_1(t) = \langle t,\ 1-t,\ 3+t^2 \rangle$$

and $$\vec{r}_2(t) = \langle 3-s,\ s-2,\ s^2 \rangle$$

intersect. What is the approximate at angle of intersection?

(a) Intersection point

Find s and t satisfying

$$t = 3-s \ ①, \quad 1-t = s-2 \ ②, \quad 3+t^2 = s^2 \ ③$$

Notice that ① and ② are the same equation so use ② and ③

$$3+t^2 = s^2 \ (i)$$
$$1-t = s-2 \ (ii)$$

solving $t=1$, $s=2$.

Point of intersection is $(1,0,4)$

To find the approximate angle of intersection,

$\vec{r}_1\,'(t) = \langle 1,-1,2 \rangle$ and $\vec{r}_2\,'(2) = \langle -1,1,4 \rangle$

$$\cos\theta = \frac{\vec{r}_1\,'(t) \cdot \vec{r}_2\,'(t)}{|\vec{r}_1\,'(t)||\vec{r}_2\,'(t)|} = \boxed{\frac{1}{\sqrt{3}}.}$$

Find $\vec{r}(t)$ if $\vec{r}'(t) = 2t\,i + 3t^2 j + \sqrt{t}\,k$ and $\vec{r}(1) = i + j$

$$\vec{r}(t) = \int \vec{r}'(t)\,dt = \int 2t\,i + 3t^2 j + t^{\frac{1}{2}} k$$

$$= \frac{2t^2}{2} i + \frac{3t^3}{3} j + \frac{2}{3} t^{\frac{3}{2}} k + c$$

$$= t^2 i + t^3 j + \frac{2}{3} t^{\frac{3}{2}} k + \vec{c}$$

To find $\vec{c}$, $\vec{r}(1) = i + j \Rightarrow$

$$i + j + \frac{2}{3} \cdot k + \vec{c} = i + j \Rightarrow \vec{c} = -\frac{2}{3} k$$

$$\Rightarrow \vec{r}(t) = t^2 i + t^3 j + \left(\frac{2}{3} t^{3/2} - \frac{2}{3}\right) k$$

This is an example of an initial value problem.