13.3 Arc Length and Curvature

In calc II, the arc length of a curve defined by parametric equations

$x = f(t)$, $y = g(t)$, $a \le t \le b$

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$$

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

For now, given a curve C defined by
$x = f(t)$, $y = g(t)$, $z = h(t)$, the length

Arc length formula

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt$$

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = \boxed{\int_a^b |\vec{r}\,'(t)|\, dt.}$$

Curvature.

A parametrization $\vec{r}(t)$ is called smooth if $\vec{r}\,'(t)$ is continuous and $\vec{r}\,'(t) \neq 0$.

Smooth means
- No sharp corners / cusps
- The tangent vector turns continuously

~~If C is a smooth curve defined by $\vec{r}(t)$~~

Recall the unit tangent vector

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|r'(t)|}$$

Curvature at a point is a measure of how quickly the curve changes direction at that point.

Curvature – magnitude of the rate of change of the unit tangent vector with respect to the arc length.

Curvature $\kappa = \left|\dfrac{dT}{ds}\right|$

Curvature computation

$$\frac{d\vec{T}}{ds} = \frac{d\vec{T}}{dt} \cdot \frac{dt}{ds} \Rightarrow \kappa = \left| \frac{dT/dt}{ds/dt} \right|$$

since $\frac{ds}{dt} = |\vec{r}\,'(t)|$ [speed is magnitude of velocity vector].

$$\kappa(t) = \frac{|T'(t)|}{|r'(t)|}$$

A more convienient formula

$$\kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|r'(t)|^3}$$

Example (Arc length and Curvature)

(a) Arc length.

Given $\vec{r}(t) = \sqrt{2}\,t\,i + e^t j + e^{-t} k \Rightarrow \vec{r}\,'(t) = \sqrt{2}\,i + e^t j - e^{-t} k$, $0 \le t \le 1$

$$|\vec{r}\,'(t)| = \sqrt{(\sqrt{2})^2 + (e^t)^2 + (-e^{-t})^2} = \sqrt{2 + e^{2t} + e^{-2t}} = \sqrt{(e^t + e^{-t})^2} = (e^t + e^{-t})$$

Recall that $(a+b)^2 = a^2 + 2ab + b^2$

Let $a = e^t$ and $b = e^{-t}$, so that

$$a^2 + 2ab + b^2 = (e^t)^2 + 2e^{-t} \cdot e^t + (e^{-t})^2$$
$$= e^{2t} + 2e^0 + e^{-2t}$$
$$= (e^t + e^{-t})^2$$

$$L = \int_0^1 |\vec{r}\,'(t)|\,dt = \int_0^1 (e^t + e^{-t})\,dt = [e^t - e^{-t}]\Big|_{t=0}^{1}$$
$$= e - e^{-1}$$

(b) Curvature

Find the curvature $\vec{r}(t) = \langle t, t^2, t^3 \rangle$ @ $(1,1,1)$

$\vec{r}\,'(t) = \langle 1, 2t, 3t^2 \rangle$, The point $(1,1,1)$ corresponds to $t=1$.

$\vec{r}\,'(1) = \langle 1,2,3 \rangle$. , $|\vec{r}\,'(t)| = \sqrt{1^2+2^2+3^2} = \sqrt{14}$.

$\vec{r}\,''(t) = \langle 0,2,6t \rangle \Rightarrow \vec{r}\,''(1) = \langle 0,2,6 \rangle$.

$$\kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|r'(t)|^3}$$

$|\vec{r}\,'(1) \times \vec{r}\,''(1)| = \langle 6,-6,2 \rangle$ so $|\vec{r}\,'(1) \times r''(1)| = \sqrt{36+36+4}$
$= \sqrt{76}$

$$\kappa(1) = \frac{|\vec{r}\,'(1) \times \vec{r}\,''(1)|}{|\vec{r}\,'(1)|^3} = \frac{\sqrt{76}}{(\sqrt{14})^3} = \frac{1}{7}\sqrt{\frac{19}{14}}.$$