Limits 14.2 and Continuity

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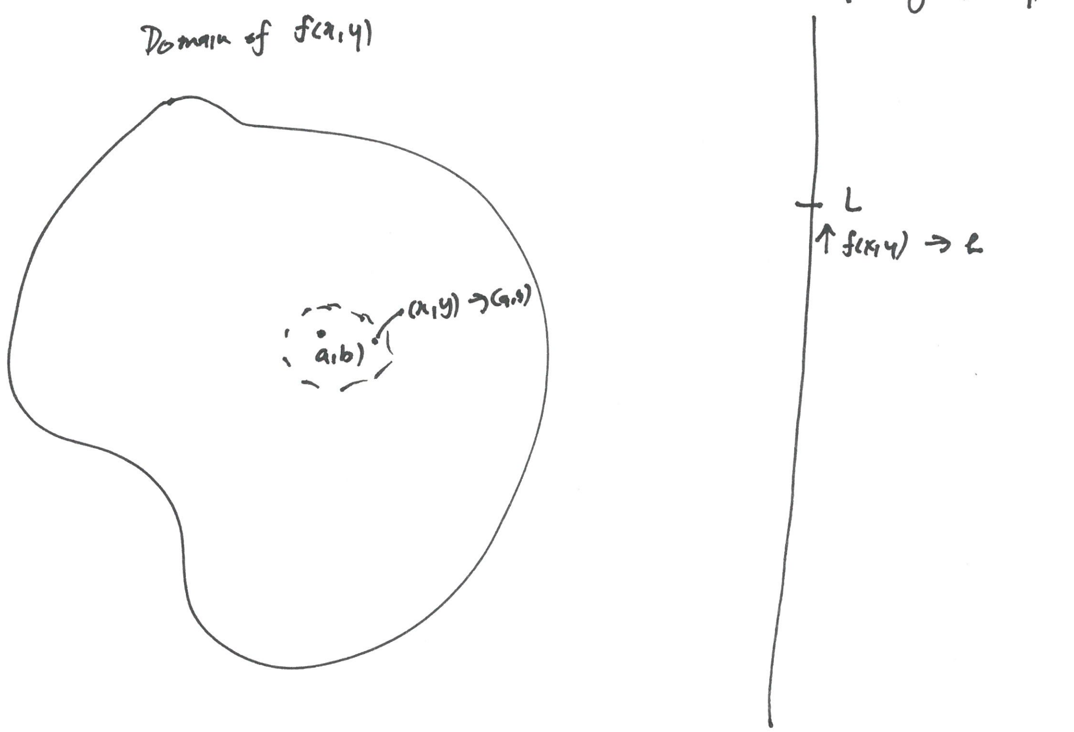
Let f be a function of two variables with a domain that includes a set of points arbitrarily close to (a,b).

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$, if for every $(x,y) \in D$ if we can make $f(x,y)$ as close to L by picking any $(x,y) \in D$ close to (a,b).

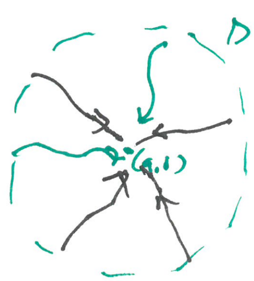
(ie) The distance between $f(x,y)$ and L, $|f(x,y) - L|$ can be made arbitrarily small by picking (x,y) such that $\sqrt{(x-a)^2 + (y-b)^2}$ is small



As in Calc I, we will start by plugging in values of $f(x,y)$ for values of x close to (a,b).

Complication

the limit has to be the same for every possible path!



There is an infinite number of paths to get to (a,b).

In Calc I, we said, the limit exists if $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$

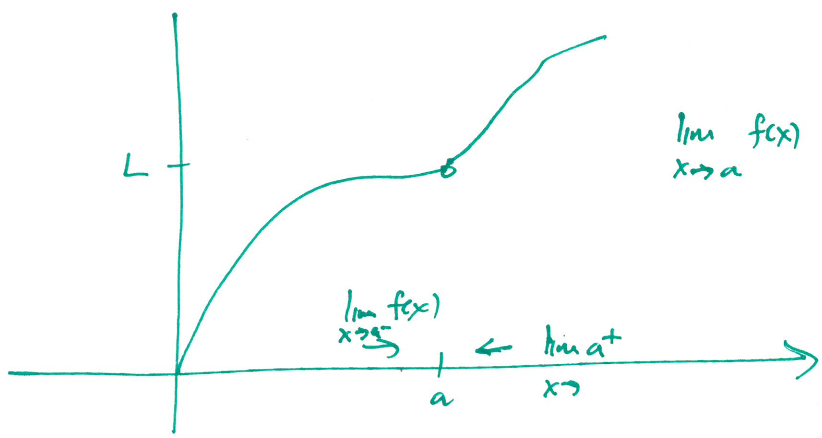
Think about climbing a mountain, there are multiple ways of getting to the top, depending on the path, the limit may "see" different.

Example #1

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$ (b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$

Notice first that $f(x,y) = \dfrac{\sin(x^2+y^2)}{x^2+y^2}$ and $g(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$ are not defined @ $(0,0)$. The function need not ~~de~~ be defined at the point

Calc I version



$\lim_{x\to a} f(x) = L$, f is not defined @ $x=a$.

Table from Matlab.

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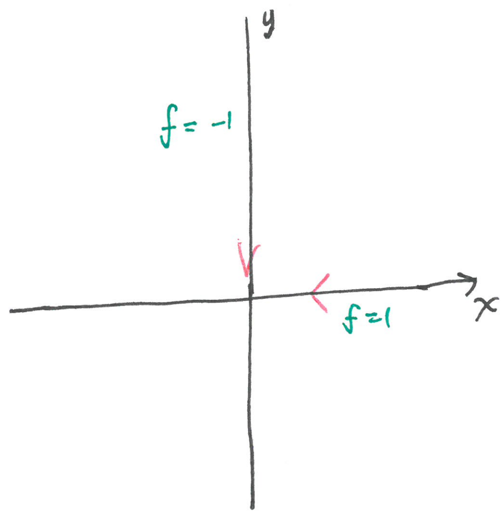
If $f(x,y) \to L_1$ as $(x,y) \to (a,b)$ along a path C_1 and $f(x,y) \to L_2$ as $(x,y) \to (a,b)$ along a path C_2, where $L_1 \neq L_2$, then

(a) $\lim_{(x,y)\to(a,b)} f(x,y)$ D.N.E.

Lets ~~first~~ show that $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ does not exist.



approach 0 along the x-axis

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=0}} \left(\frac{x^2-y^2}{x^2+y^2}\right) \quad \cancel{\neq \lim_{x\to 0}}$$

$f(x,0) = \frac{x^2}{x^2} = 1$ for all $x \neq 0$ so $\lim_{\substack{(x,y)\to(0,0)\\ \text{along } x\text{-axis}}} f(x,y) = 1$

approach 0 along y-axis

set $x=0$, $f(0,y) = \frac{-y^2}{y^2} = -1$ for $y \neq 0 \Rightarrow \lim_{\substack{(x,y)\to(0,0)\\ \text{along } y\text{-axis}}} f(x,y) = -1.$

Example #2

$f(x,y) = \frac{xy}{x^2+y^2}$, $\lim_{(x,y)\to(0,0)} f(x,y)$ exists?

If $y=0$, $f(x,0) = \frac{0}{x^2} \Rightarrow \lim_{\substack{(x,y)\to(0,0)\\ \text{along } x\text{-axis}}} = 0$

If $x=0$, $f(0,y) = \frac{0}{y^2} = 0 \Rightarrow \lim_{(x,y)\to(0,0)} f(x,y) = 0$

Is this enough?,

No how about along $y=x$?

$$f(x,x) = \frac{x^2}{x^2+x^2} = \frac{1}{2} \quad \text{so} \quad f(x,y) \to \frac{1}{2} \quad \text{as} \quad (x,y)\to(0,0)$$

so the limit does not exist.

Example #3

$$f(x,y) = \frac{xy^2}{x^2+y^4}$$, how about along $y=mx$.

$$f(x,y) = f(x,mx) = \frac{x(mx)^2}{x^2+(mx)^4} = \frac{m^2x^3}{x^2+m^4x^4} = \frac{m^2x}{1+m^4x^2}$$

as $(x,y)\to 0$ $\frac{m^2x}{1+m^4x^2} \to 0$, so one might be tempted

to say

$$\lim_{(x,y)\to(0,0)} \left(\frac{xy^2}{x^2+y^4}\right) = 0$$

However, let $x=y^2$

$$f(y^2,y) = \frac{y^2\cdot y^2}{(y^2)^2+y^4} = \frac{y^4}{2y^4} = \frac{1}{2}$$ so the limit does

not exist.

Showing existence of a limit in 3D is a tough task, we have to go to the classical definition

$$\boxed{\lim_{(x,y)\to(a,b)} f(x,y) = L \text{ if for every number } (\varepsilon > 0) \text{ (small), there exists a number } \delta > 0 \text{ such that if } (x,y) \in D \text{ and } 0 < \sqrt{(x-a)^2+(y-b)^2} < \delta \text{ then } |f(x,y) - L| < \varepsilon.}$$

for some special cases we can use "tricks"

eg $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$

Let $r^2 = x^2+y^2$. (Polar coordinates)

$$\lim_{r\to 0^+} \frac{\sin(r^2)}{r^2} \overset{H}{=} \lim_{r\to 0^+} \frac{2r\cos(r^2)}{2r} = \lim_{r\to 0^+} \cos(r^2) = 1.$$

OR
the functions are continuous as in Calc I, we can just directly substitute.

A function f of 2 variables is called continuous at (a,b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

~~H~~ • Continuous functions have surfaces with no holes or breaks.

Example Polynomials:

$\displaystyle\lim_{(x,y)\to(1,2)} 5x^3 - x^2y^2$, $5x^3 - x^2y^2$ is continuous so $\displaystyle\lim_{(x,y)\to(1,1)} (5x^3 - x^2y^2) = 5(1^3) - 1^2(2^2) = 1.$

Example

① $$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}$$

$$\frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} \cdot \frac{\sqrt{x^2+y^2+1}+1}{\sqrt{x^2+y^2+1}+1}$$

$$\frac{(x^2+y^2)\left(\sqrt{x^2+y^2+1}-1\right)}{(x^2+y^2+1)-1} = \frac{\cancel{(x^2+y^2)}\cdot\left(\sqrt{x^2+y^2+1}-1\right)}{\cancel{(x^2+y^2)}}$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} = \lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2+1}-1 = \underline{0}$$

~~$\lim_{(x,y)\to(0,0)}$~~

② $$\lim_{(x,y)\to(1,0)} \ln\left(\frac{1+y^2}{x^2+xy}\right) = \ln(1) = 0. \quad \text{continuous} \checkmark.$$

③ $$\lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)(x^2-y^2)}{x^2+y^2} = \lim_{(x,y)\to(0,0)} (x^2-y^2) = 0.$$

④ $$\lim_{(x,y)\to(0,0)} \frac{(x^2\sin^2 y)}{x^2+2y^2}$$

$$\frac{x^2}{x^2+2y^2} < 1$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2\sin^2(0)}{x^2+2(0)} = 0.$$

along $x=0$

$$0 \le \frac{x^2\sin^2(y)}{x^2+2y^2} \le \sin^2(y)$$

$0 \rightsquigarrow 0$ $\qquad$ $\sin^2(y) \rightsquigarrow 0$

by Squeeze Theorem.

Continuity - Examples of continuous function

Polynomials - $cx^m y^m$

eg $f(x,y) = x^5 + x^4 y^3 + x^2.$

Rational functions - Ratio of polynomials are continuous on their domain.

- All polynomials are continuous in $\mathbb{R}^2$.

Examples

1. $F(x,y) = \dfrac{1 + x^2 + y^2}{1 - x^2 - y^2}$

is not continuous on $1 - x^2 - y^2 = 0$, since it is not defined there

$F(x,y)$ is continuous on

$$D = \{(x,y) \mid x^2 + y^2 \neq 1\}.$$

Remark

If we want to compute a limit in D, we simply plug in

eg $\displaystyle\lim_{(x,y)\to(0,0)} \left(\frac{1 + x^2 + y^2}{1 - x^2 - y^2}\right) = 1.$

2. $F(x,y) = \ln(x^2 + y^2 - 4)$

Continuous on $D = \{(x,y) \mid x^2 + y^2 > 4\}$

$$\lim_{(x,y)\to(2,2)} = \ln\left(\lim_{(x,y)\to(2,2)} (x^2 + y^2 - 4)\right)$$

$$= \ln\left(2^2 + 2^2 - 4\right)$$

$$= \ln(2^2)$$