14.3 # Partial Derivatives

Let f be a function of two variables, $z = f(x,y)$

- Suppose we let x vary, but keep y fixed, say $y=b$, then the function $z = f(x,b)$ becomes a function of one variable.

$$g(x) = f(x,b)$$

Def

If g has a derivative at $x=a$, we call the partial derivative of f w.r.t x.

$$f_x(a,b) = g'(a)$$

$$g'(a) = \lim_{h\to 0} \frac{g(a+h) - g(a)}{h}$$

so that

$$f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

similarly

$$f_y(a,b) = \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h}$$

In general

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}.$$

Notation

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = D_x f$$

$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = D_y f.$$

Computing Partial derivatives

1. $f_x(x,y)$ – treat y as a constant and differentiate $f(x,y)$ with respect to x

2. $f_y(x,y)$ – treat x as a constant and differentiate $f(x,y)$ w.r.t to y.

Example #1

$$f(x,y) = y^5 - 3xy$$

(i) $f_x(x,y) = \frac{d}{dx}\left(y^5 - 3xy\right)$

$= 0 - 3y$

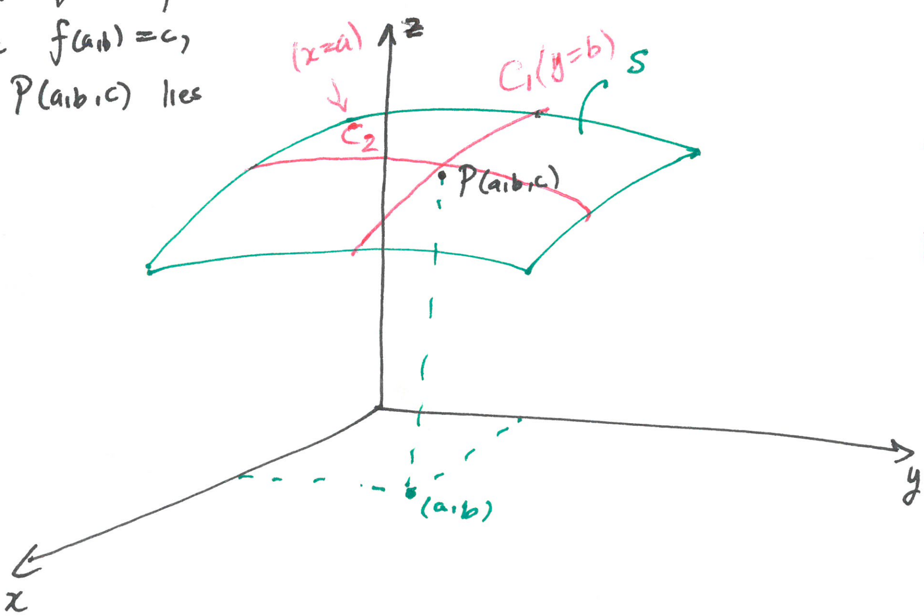
(ii) $f_y(x,y) = \frac{d}{dy}\left(y^5 - 3xy\right)$

$= 5y^4 - 3x.$

Interpretation of Partial Derivatives

$z = f(x,y)$ represents a surface (S) in the xy plane

Suppose $f(a,b) = c$, then $P(a,b,c)$ lies on S



If we fix $y=b$, we restrict ourselves to the curve C_1 $g_1(x) = f(x,b)$

If we fix $x=a$, we restrict ourselves to the curve C_2 $g_2(y) = f(a,y)$

Along C_1 $g_1'(a) = f_x(a,b)$ at P

Along C_2 $g_2'(b) = f_y(a,b)$ at P

Partial derivatives are the slopes of the tangent lines to C_1 and C_2!

Example

$$f(x,y) = \sqrt{4 - x^2 - 4y^2}$$

$f_x(1,0)$

$$f_x(x,y) = \frac{1}{2}\left(4 - x^2 - 4y^2\right)^{-\frac{1}{2}} \cdot (-2x)$$

$$f_x(1,0) = \frac{1}{2}\left(4 - 1^2 - 4 \cdot 0^2\right)^{-\frac{1}{2}}(-2)$$

$$= \frac{-1}{\sqrt{3}}$$

Slope of the tangent line to intersection of $y=0$ and $f(x,y)$

$$f_y(x,y) = \frac{1}{2}\left(4 - x^2 - 4y^2\right)^{-\frac{1}{2}} \cdot (-8y)$$

$$f_y(1,0) = \frac{1}{2}\left(4 - 1^2 - 4 \cdot 0^2\right)^{-1/2} \cdot (-8 \cdot 0)$$

$$= 0.$$

Slope of tangent line to intersection of $x=1$ and $f(x,y)$.

Functions of Two or More Variables

$$f(x,y,z) = x \sin(y - z)$$

$f_x(x,y,z) = \sin(y - z)$ [hold y and z constant].

$f_y(x,y,z) = x \cos(y - z) \cdot 1$. [hold x and z constant].

Higher Derivatives

1. $(f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$
2. $(f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$
3. $(f_y)_x = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$
4. $(f_y)_y = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$

NOTE:

f_{xy} – x, then y

f_{yx} – y, then x.

Clairaut's Theorem

Suppose f is defined and continuous on a disk D, that contains (a,b), then

$$f_{xy}(a,b) = f_{yx}(a,b)$$

$\boxed{\text{Higher order partial derivatives can also be computed}}$

e.g. $\quad f_{yxx} = (f_{yx})_x = \dfrac{\partial}{\partial x}\left(\dfrac{\partial^2 f}{\partial x \partial y}\right) = \dfrac{\partial^3 f}{\partial x^2 \partial y}$

Example #1

$$u = \ln(x+2y)$$

$u_{xy} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial x}\right) \Rightarrow$ compute $\dfrac{\partial u}{\partial x}$ first

$$\frac{\partial u}{\partial x} = \frac{1}{x+2y} = (x+2y)^{-1}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\left[(x+2y)^{-1}\right] = -1(x+2y)^{-2}\cdot(2)$$

$$= \frac{-2}{(x+2y)^2}.$$

$u_{yx} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial y}\right) \Rightarrow$ compute $\dfrac{\partial u}{\partial y}$ first.

$$\frac{\partial u}{\partial y} = \frac{1}{x+2y}(2) = \frac{2}{x+2y}$$

$$u_{yx} = \frac{\partial}{\partial x}\left(\frac{2}{x+2y}\right) = \frac{\partial}{\partial x}\left[2(x+2y)^{-1}\right]$$

$$= 2\cdot(-1)(x+2y)^{-2}\cdot 1 = \frac{-2}{(x+2y)^2}$$

So we conclude $u_{yx} = u_{xy}$.

Higher order derivatives

$$f(x,y,z) = \sin(3x + yz)$$

① $f_x(x,y,z) = \cos(3x + yz) \cdot \frac{\partial}{\partial x}(3x + yz)$

$$= \cos(3x + yz) \cdot 3$$

② $f_{xx} = \frac{\partial}{\partial x}(f_x) = \frac{\partial}{\partial x}\left(3\cos(3x + yz)\right)$

$$= -9\sin(3x + yz)$$

③ $f_{xxy} = \frac{\partial}{\partial y}\left(f_{xx}\right) = \frac{\partial}{\partial y}\left(-9\sin(3x + yz)\right)$

$$= -9\cos(3x + yz) \cdot \frac{\partial}{\partial y}\left(3x + yz\right)$$

$$= -9\cos(3x + yz)(z)$$

$$= -9z\cos(3x + yz).$$

Partial differential Equations

(1) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{in } (2D) \quad \text{or} \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad \text{in } \underline{3D}$$

is called Laplace's equation.

- Solutions to this equation are called harmonic functions.

Laplace's equation has important applications in heat conduction, fluid flow and electric potential.

$u(x,y) = e^x \sin(y)$ is an example of a harmonic function

Check!

$u_x = e^x \sin(y)$ $\qquad$ $u_y = e^x \cos(y)$

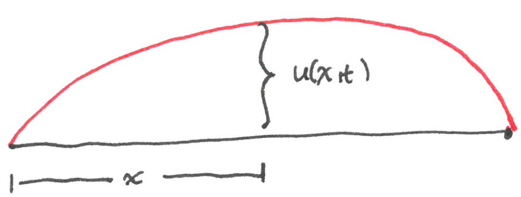
$u_{xx} = e^x \sin(y)$ $\qquad$ $e_{yy} = -e^x \sin(y)$

so $\quad u_{xx} + u_{yy} = e^x \sin(y) - e^x \sin(y) = 0.$

(2) Wave Equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

$u(x,t)$ - displacement of of a vibrating string @ time t and a distance x from the end of the string



a is a constant that depends on the density of the string

$u(x,t) = \sin(x - at).$ is a solution

Check

$u_x = \cos(x - at)$ $\qquad$ $u_t = -a\cos(x - at)$

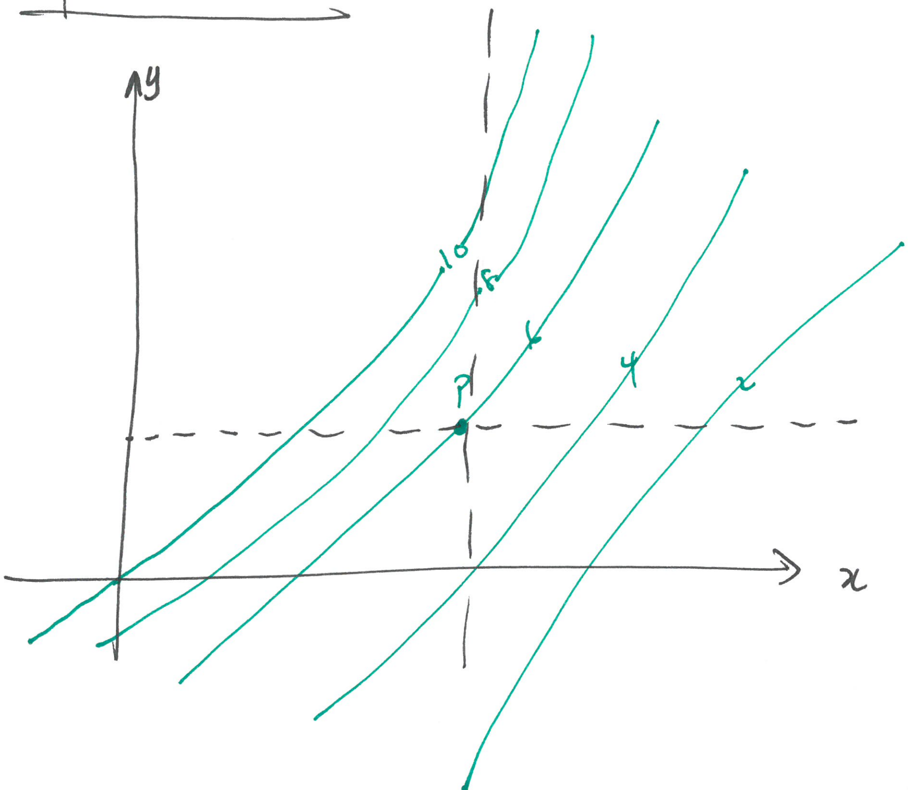
$u_{xx} = -\sin(x - at)$ $\qquad$ $u_{tt} = -a^2 \sin(x - at) = a^2 u_{xx}.$

14.3

If f_{xy} and f_{yx} are continuous on a disk D containing (a,b), then

$$f_{xy}(a,b) = f_{yx}(a,b).$$

Graphical intuition



1. $f_x(P)$ - If we fix y and allow x to vary $f\downarrow$ as $x\uparrow$ $f_x(P) < 0$

2. $f_y(P) > 0$, ~~fy~~ $f\uparrow$ as $y\uparrow$ for a fixed x

3 $f_{xx}(P) = \frac{\partial}{\partial x}(f_x)$, fix y and allow x to vary

Note that Points to the right of P are spaced further apart, f decreases less quickly $f_{xx} > 0$

4. $f_{xy}(P) = \frac{\partial}{\partial y}(f_x)$ - fix x and allow y to vary, f_{xy} is the rate of change of f_x, as y increases level curves are closer together in the $+x$ direction as y increases so f_x decreases more quickly w.r.t x for y values above P. $f_{xy}(P) < 0$

$f_{yy}(P) > 0$

5. $f_{yy}(P) = \frac{\partial}{\partial y}(f_y)$ - fix x and allow y to vary, f_{yy} is the rate of change of f_y as y increases. f increases faster w.r.t y on Points above P

so $f_{yy}(P) > 0$