(14.5) The Chain Rule

Recall the 1D chain Rule

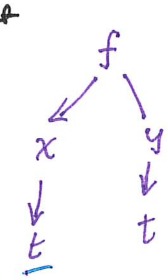
If $y = f(x)$ and $x = g(t)$

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

3D version I

If $z = f(x,y)$ and $x = g(t)$ and $y = h(t)$ so that

$$z = f(g(t), h(t))$$



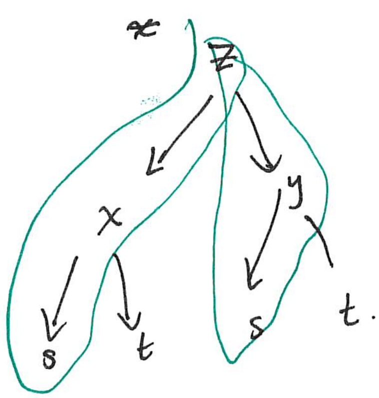
then $$\frac{dz}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

Version II

If $z = f(x,y)$ and $x = g(s,t)$ and $y = h(s,t)$, then

$$\frac{dz}{ds} = \frac{dz}{dx} \cdot \frac{dx}{ds} + \frac{dz}{dy} \frac{dy}{ds}$$

$$\frac{dz}{dt} = \frac{dz}{dx} \cdot \frac{dx}{dt} + \frac{dz}{dy} \cdot \frac{dy}{dt}$$



$$\frac{dz}{ds}$$

Example #1

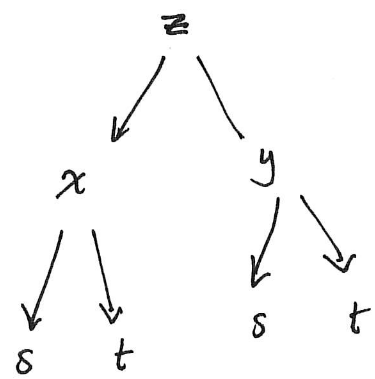
(a) $z = \cos(x+4y)$, $x = 5t^4$, $y = \frac{1}{t}$

$$\frac{dz}{dt} = \frac{df}{dx} \cdot \frac{dx}{dt} + \frac{df}{dy} \cdot \frac{dy}{dt}$$

$$\frac{dz}{dt} = -\sin(x+4y) \cdot 1 \cdot 20t^3 + -\sin(x+4y) \cdot 4 \cdot -t^{-2}$$

$$= -20t^3 \sin(x+4y) + \sin(x+4y) \cdot 4t^{-2}$$

(b) $z = x^2 y^3$, $x = s\cos t$, $y = s\sin t$



$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{dy}{ds}$$

$$= 2xy^3 \cdot \cos t + 3y^2x^2 \sin t.$$

$$\frac{dz}{dt} = \frac{dz}{dx} \cdot \frac{dx}{dt} + \frac{dz}{dy} \cdot \frac{dy}{dt}$$

$$= 2xy^3(-s\sin t) + 3y^2x^2(-s\cos t)$$

Example

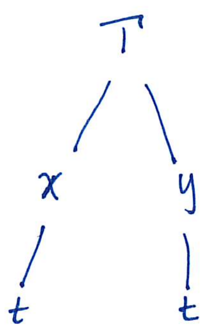
The temperature at a point (x, y) is $T(x,y)$ in °C.

The position of a bug is given by $x = \sqrt{1+t}$, $y = 2 + \frac{1}{3}t$

$T_x(2,3) = 4$ and $T_y(2,3) = 3.$

How fast is the temperature rising on the bug's path after 3s.

$\frac{dT}{dt}$ – Rate of increase of T w.r.t time



$$\boxed{\frac{dT}{dt} = \frac{dT}{dx} \cdot \frac{dx}{dt} + \frac{dT}{dy} \cdot \frac{dy}{dt}}$$

$$\frac{dT}{dt} = \frac{\partial T}{dx} \cdot \frac{dx}{dt} + \frac{\partial T}{dy} \cdot \frac{dy}{dt}$$

@ $t=3$, $x = \sqrt{1+3} = 2$, $y = 2 + \frac{1}{3} \cdot 3 = \underline{3}$

$$\left.\frac{dT}{dt}\right|_{t=3} = T_x(2,3) \cdot \frac{d}{dt}\left((1+t)^{\frac{1}{2}}\right) + T_y(2,3) \frac{d}{dt}\left(2 + \frac{1}{3}t\right)$$

$$= 4 \cdot \left[\frac{1}{2}(1+t)^{-\frac{1}{2}}\right] + 3\left[\frac{1}{3}\right]\Bigg|_{t=3}$$

$$= \frac{4}{2}\frac{1}{\sqrt{4}} + 3 \cdot \frac{1}{3}$$

$$= \frac{4}{4} + \frac{3}{3} = \underline{2\,°C/s}$$

The temperature is increasing at a rate of 2°C/second

Implicit differentiation

Suppose $F(x,y)=0$, y defined implicitly as a function of x $[y=f(x)]$.
Find $\frac{dy}{dx}$ $\quad$ (x is independent)

[Chain Rule]

$$\begin{array}{ccc} & F & \\ \swarrow & & \searrow \\ x & & y \\ & & \downarrow \\ & & x \end{array}$$

$$\frac{\partial F}{\partial x}\cdot\frac{dx}{dx}+\frac{\partial F}{\partial y}\cdot\frac{dy}{dx}=0$$, solving for $\frac{dy}{dx}$ $\left(\frac{dx}{dx}\equiv 1\right)$

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$$

Example #1.

$y\cos x = x^2+y^2$, find $\frac{dy}{dx}$. , first write in the form $F(x,y)=0$.

$$F(x,y) = y\cos x - x^2 - y^2.$$

$$F_x(x,y) = -y\sin x - 2x, \qquad F_y(x,y) = \cos(x) - 2y$$

$$\frac{dy}{dx} = \frac{-F_x}{F_y} = \frac{-(y\sin(x) - 2x)}{\cos(x)-2y} = \frac{y\sin(x)+2x}{\cos(x)-2y}.$$

Suppose $F(x,y,z)=0$, $z=f(x,y)$ $\Rightarrow$ $F(x,y,f(x,y))=0$ [x and y are independent]

$$\begin{array}{ccc} & F & \\ \swarrow & \downarrow & \searrow \\ x & y & z \end{array}$$

$$\frac{\partial F}{\partial x}\cdot\frac{dx}{dx}+\frac{\partial F}{\partial y}\cdot\frac{dy}{dx}+\frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial x}=0$$, since $\frac{dx}{dx}=1$ and $\frac{\partial}{\partial x}(y)=0$

$$\frac{\partial F}{\partial x}+\frac{\partial F}{\partial z}\cdot\frac{dz}{dx}=0$$

$$\frac{\partial z}{\partial x} = \frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = \frac{-F_x}{F_z}.$$

Similarly, $$\frac{\partial z}{\partial y} = \frac{-\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = \frac{-F_y}{F_z}.$$

Example #2

$$x^2 + 2y^2 + 3z^2 = 1$$

$$F(x,y,z) = x^2 + 2y^2 + 3z^2 - 1$$

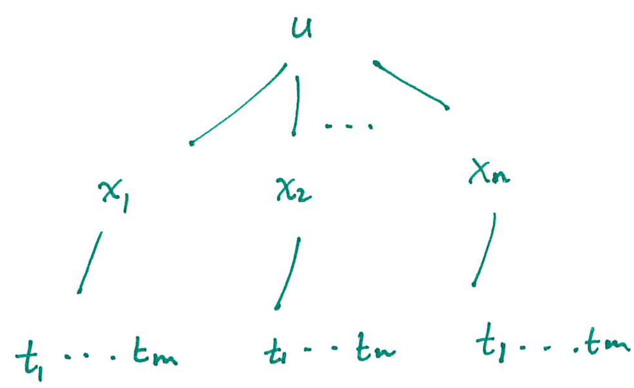
$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} = -\frac{2x}{6z}$$

$$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = -\frac{2 \cdot 2y}{6z} = \frac{-4y}{6z}.$$

14.6

General version of the chain rule (Explicit)

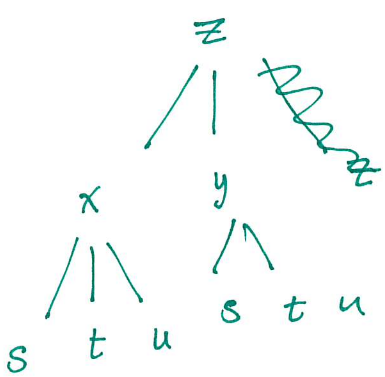
Suppose u is a function of n variables $x_1, x_2, \ldots, x_n$ and each x_i is a function of $t_1, t_2, \ldots, t_m$ i.e.



$$\frac{du}{dt_i} = \frac{\partial u}{\partial x_1}\cdot\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\cdot\frac{\partial x_2}{\partial t_i} + \frac{\partial u}{\partial x_3}\cdot\frac{\partial x_3}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\cdot\frac{\partial x_n}{\partial t_i}$$

Example

$$z = x^4 + x^2 y \qquad x = s + 2t - u, \quad y = stu^2$$



$$\cancel{\frac{\partial z}{\partial s} = \frac{\partial x}{\partial s}\cdot\frac{\partial s}{\partial}}$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}$$

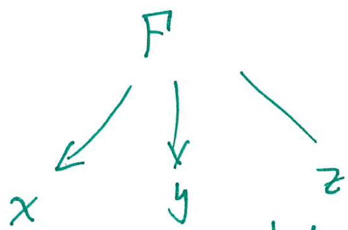
$$= (4x^3 + 2xy)\cdot 1 + x^2\cdot tu^2$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}.$$

14.5 Implicit function theorem

e.g $\quad x^2 - y^2 + z^2 - 2z = 4$

$F(x,y,z) = 0 \Rightarrow F(x,y,z) = x^2 - y^2 + z^2 - 2z - 4$



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$$\frac{\partial F}{\partial y}\cdot\frac{dy}{dy} + \frac{\partial F}{\partial x}\cdot\frac{\partial x}{\partial y} + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial y} = 0$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

$$\left.\begin{array}{l} F_x(x,y,z) = 2x \\ F_y = -2y \end{array}\right\} \quad \frac{\partial z}{\partial y} = \frac{-2y}{2x} = \underline{\frac{-y}{x}}$$