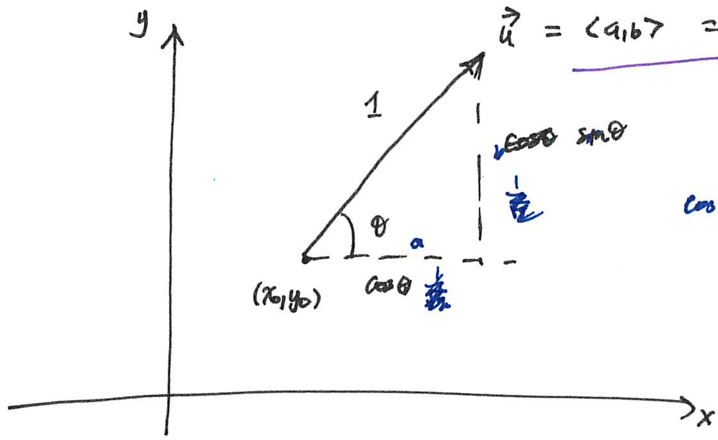
(14.6) Directional Derivatives

For $z = f(x,y)$, $f_x(x_0,y_0)$ and $f_y(x_0,y_0)$ represent rates of change of z in the x and y directions.

$\vec{u} = \langle a,b \rangle = \langle \sin\theta, \cos\theta \rangle$



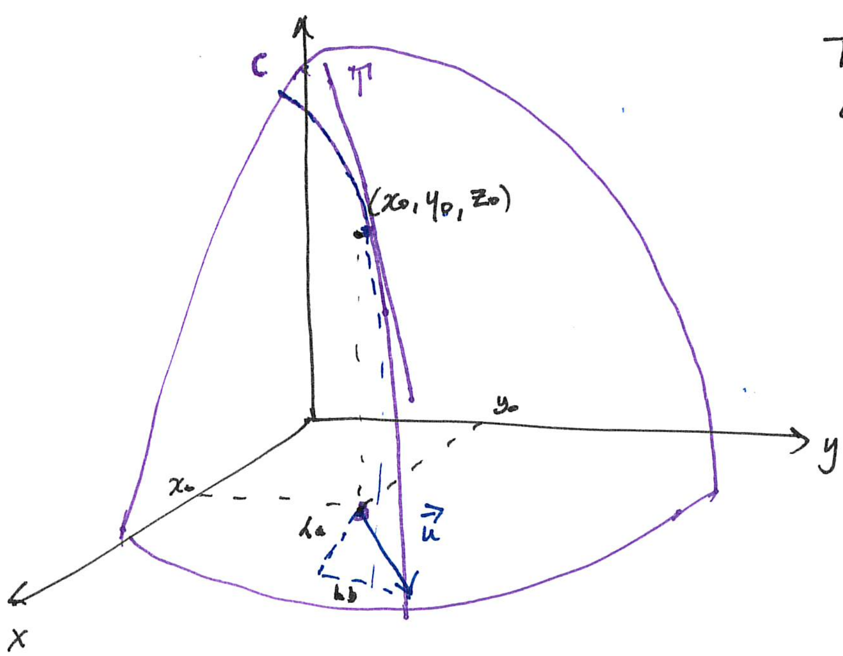
$\cos\theta = \frac{a}{1}$ $\quad$ $\sin\theta = \frac{b}{1}$

$a = \cos\theta$

$b = \sin\theta$

Sup Problem

Find the ~~dir~~ derivative in the direction $\vec{u} = \langle a,b \rangle$. [$\vec{u}$ is a unit vector]



The slope of T is the rate of change of z in the direction $\vec{u}$.

Directional derivative of f @ (x_0,y_0) in the direction $\vec{u}$

$$D_u f(x_0,y_0) = \lim_{h \to 0} \frac{f(x_0+ha, y_0+hb) - f(x_0,y_0)}{h}$$

14.6. ① If f is differentiable, then the directional derivative in the direction of the unit vector $u = \langle a, b \rangle$ is

FACT

$$D_u f(x,y) = f_x(x,y)\, a + f_y(x,y)\, b$$

② If u makes an angle θ with the positive x-axis then $u = \langle \cos\theta, \sin\theta \rangle$ so that

$$D_u f(x,y) = f_x(x,y)\cos\theta + f_y(x,y)\sin\theta .$$

Example #1.

Find the directional derivative of f at $(1,1)$ in the direction $\theta = \frac{\pi}{6}$ of

$$f(x,y) = x^3y^4 + x^4y^3.$$

$$f_x(x,y) = 3x^2y^4 + 4x^3y^3$$
$$f_y(x,y) = x^3 4y^3 + x^4 3y^2$$

$$\begin{aligned} D_u f(x,y) &= f_x(x,y)\cos\theta + f_y(x,y)\sin\theta \\ &= (3x^2y^4 + 4x^3y^3)\cos\theta + x^3 4y^3 + x^4 3y^2 \sin\theta \\ &= 3x^2y^4 + 4x^3y^3 \cos\left(\frac{\pi}{6}\right) + x^3 4y^3 + x^4 3y^2 \sin\left(\frac{\pi}{6}\right) \end{aligned}$$

$$D_u f(1,1) = \cancel{3x^2y^4 + 4x^3y^3 \tfrac{\sqrt{3}}{2} +}\ f_x(1,1)\cos\left(\frac{\pi}{6}\right) + f_y(1,1)\sin(\pi/6) = 7 \cdot \frac{\sqrt{3}}{2} + 7 \cdot \frac{1}{2}$$

GRADIENT VECTOR

Notice that

$$\begin{aligned} D_{\vec u} f(x,y) &= f_x(x,y)\, a + f_y(x,y)\, b \\ &= \cancel{\langle f_x(x,y) \rangle} \\ &\quad \langle f_x(x,y), f_y(x,y) \rangle \cdot \langle a, b \rangle . \end{aligned}$$

The vector

$\langle f_x(x,y), f_y(x,y) \rangle$ is called the gradient of f.

Notation

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j$$

So

$$\boxed{D_{\vec u} f(x,y) = \nabla f(x,y) \cdot \vec{u} .}$$

Example

Find the directional derivative of $f(x,y) = \dfrac{x}{x^2+y^2}$ at $(1,2)$ also in the direction $\vec{v} = \langle 3,5\rangle$.

$$\boxed{D_{\vec{u}} f(x,y) = \nabla f(x,y)\cdot\vec{u}.}$$

make $\vec{u}$ unit vector!

$$\nabla f(x,y) = \langle f_x, f_y\rangle \qquad \vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{\langle 3,5\rangle}{\sqrt{3^2+5^2}} = \frac{\langle 3,5\rangle}{\sqrt{34}}$$

$$f_x(x,y) = \frac{(x^2+y^2)\frac{d}{dx}(x) - x\cdot\frac{d}{dx}(x^2+y^2)}{(x^2+y^2)^2}$$

$$= \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \Rightarrow f_x(1,2) = \frac{2^2-1^2}{(1^2+2^2)^2} = \frac{3}{25}$$

$$f_y(x,y) = \frac{(x^2+y^2)\cdot\frac{d}{dy}(x) - x\cdot\frac{d}{dy}(x^2+y^2)}{(x^2+y^2)^2}$$

$$= \frac{-2xy}{(x^2+y^2)^2} \Rightarrow f_y(1,2) = \frac{-2\cdot1\cdot2}{(1^2+2^2)^2} = \frac{-4}{25}.$$

$$D_{\vec{u}} f(1,2) = \left\langle \frac{3}{25}, \frac{-4}{25}\right\rangle\cdot\left\langle \frac{3}{\sqrt{34}}, \frac{5}{\sqrt{34}}\right\rangle \cdot \cancel{\frac{1}{5}} = \frac{-11}{25\sqrt{34}}.$$

$$= \left\langle \frac{3}{25}, \frac{-4}{25}\right\rangle\cdot\left\langle \frac{3}{5}, 1\right\rangle$$

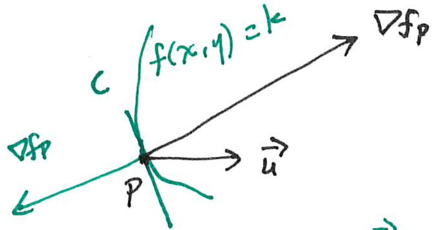
$$= \frac{9}{75} - \frac{4}{25}$$

The Directional derivative in any direction $\vec{v}$ is

$$D_{\vec{v}} f(x,y) = \nabla f(x,y)\cdot\frac{\vec{v}}{\|\vec{v}\|}$$

14.6 **Interpretation of the gradient vector** Let $\nabla f(x,y,z) = \nabla f_P$

Given $z = f(x,y)$ at $P = (x_0, y_0, z_0)$.



Recall that for any vectors $\vec{u}$ and $\vec{v}$, $\vec{v} \cdot \vec{u} = \|\vec{v}\| \, \|\vec{u}\| \cos\theta$, θ is the angle between $\vec{v}$ and $\vec{u}$.

$$D_u f(P) = \nabla f_P \cdot \vec{u} = \|\nabla f_P\| \cos\theta$$

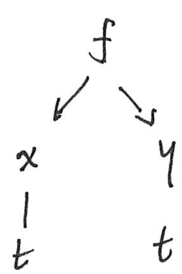
$\cos\theta$ has maximum value @ $\theta = 0$, so $D_u f(P)$ is largest when $\theta = 0$.

i.e when u points in the same direction as ∇f_P

- ∇f_P points in the direction of maximum rate of increase
- f decreases most rapidly in the opposite direction. $-\nabla f_P$.

Suppose P lies on $f(x,y) = k$ (level curve)

We can parameterize the curve C by $\langle x(t), y(t) \rangle$

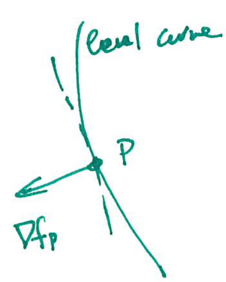


$f(x,y) = k$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} = 0$$

$$= \nabla f_P \cdot \vec{c}\,'(t) = 0$$

This means that $\nabla f_P \perp$ the level curve



14.6 #46

Let $f(x,y) = \sin(xy)$ ~~at (1,0)~~

Find the maximum rate of change of f @ $(1,0)$ and the direction in which it occurs.

$$\nabla f(x,y) = \langle +y\cos(xy),\ +x\cos(xy) \rangle$$

$$\nabla f(1,0) = \langle 0, 1 \rangle$$

maximum rate of change $= |\nabla f| = |\langle 0,1 \rangle|$

$= 1$

direction of maximum rate of change is $\nabla f = \langle 0,1 \rangle$.

14.6

Functions of 3 or more variables

Let $f(x,y,z)$ be differentiable and $\vec{u} = \langle a,b,c \rangle$, then

$$D_{\vec{u}} f(x,y,z) = f_x(x,y,z)\, a + f_y(x,y,z)\, b + f_z(x,y,z)\, c$$

Since the gradient of $f(x,y,z)$ is defined

$$\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z) \rangle$$

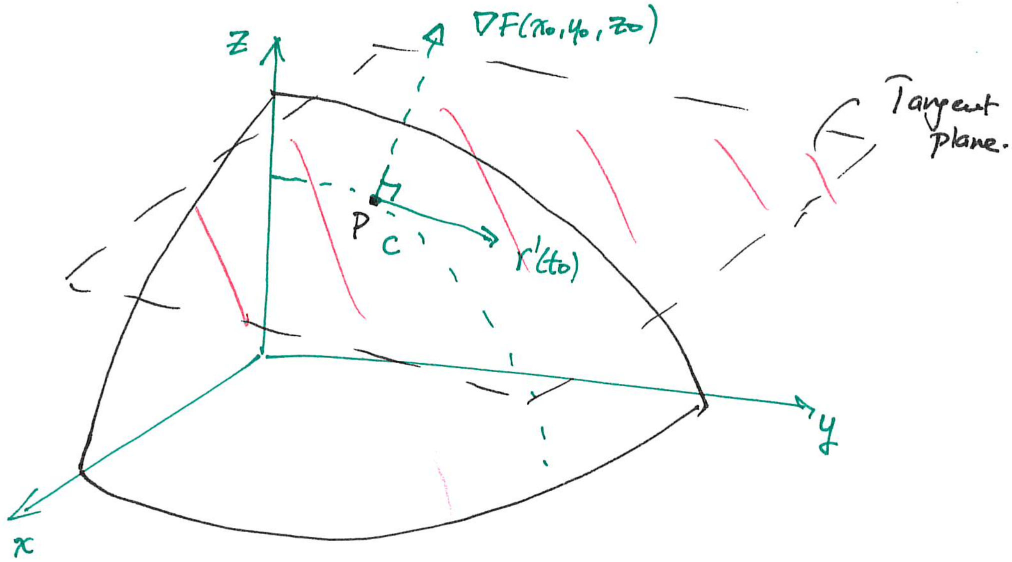
$$D_{\vec{u}} f(x,y,z) = \nabla f(x,y,z) \cdot \vec{u}.$$

14.6 Tangent planes and level surfaces.

Suppose a surface S is defined implicitly by $F(x,y,z)=k$.
i.e the level curve of $F(x,y,z)$.

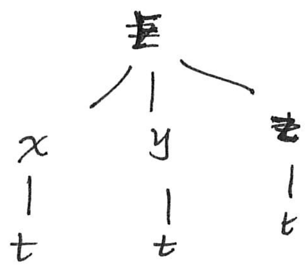
- $P=(x_0,y_0,z_0)$ is a point on S.

Let c be any curve on S



$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$.

$F(x(t), y(t), z(t)) = k$



$$\frac{\partial F}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial F}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial F}{\partial z}\cdot\frac{dz}{dt} = 0$$

$$\langle F_x, F_y, F_z \rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle = 0$$

$$\nabla F(x_0,y_0,z_0)\cdot r'(t_0) = 0$$

$\nabla F(x_0,y_0,z_0) \perp \vec{r}'(t_0)$ for any curve ~~pt~~ passing ~~thro~~ through P.

So if $\nabla F(x_0,y_0,z_0) \neq 0$ it is the normal vector to the level surface $F(x,y,z)=k$ @ $P(x_0,y_0,z_0)$

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$$F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0) = 0.$$

Normal line

- line perpendicular to the tangent plane.

$$\vec{r}(t) = \vec{r_0} + t\vec{v}$$

$$\vec{r}(t) = \langle x_0, y_0, z_0\rangle + t\langle F_x(x_0,y_0,z_0),\ F_y(x_0,y_0,z_0),\ F_z(x_0,y_0,z_0)\rangle$$

Parametric Equations: ~~r(t) = x0~~

$$x = x_0 + t\,F_x(x_0,y_0,z_0)$$
$$y = y_0 + t\,F_x(x_0,y_0,z_0)$$
$$z = z_0 + t\,F_y(x_0,y_0,z_0)$$

$$\frac{x-x_0}{F_x(x_0,y_0,z_0)} = \frac{y-y_0}{F_y(x_0,y_0,z_0)} = \frac{z-z_0}{F_y(x_0,y_0,z_0)}$$ [Symmetric form].

Example

Find the equation of the tangent plane to the surface

$$4x^2 + 9y^2 - z^2 = 16 \quad @ \quad P = (2,1,3)$$

$$F(x,y,z) = 4x^2 + 9y^2 - z^2 - 16$$

$$\nabla F(x,y,z) = \langle 8x,\ 18y,\ -2z\rangle$$

$$\nabla F(2,1,3) = \langle 16,\ 18,\ -6\rangle$$

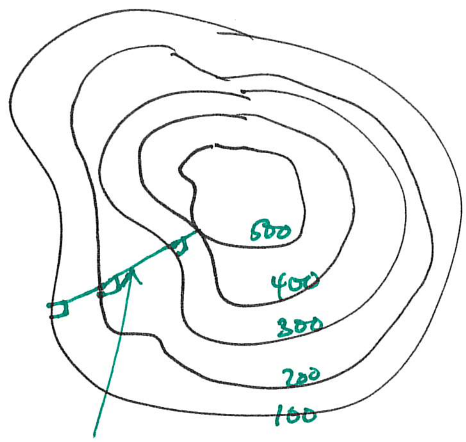
Equation of Tangent ~~line~~: plane.

$$16(x-2) + 18(y-1) - 6(z-3) = 0.$$

OR

$$16x + 8y - 6z = 32.$$

14.6 Graphical intuition



Curve of steepest decent.

Example [Normal line].

$$\vec{r}(t) = \vec{r}_0 + t\vec{v}$$

$$\langle 2, 1, 3 \rangle + t \langle 16, 18, -6 \rangle.$$