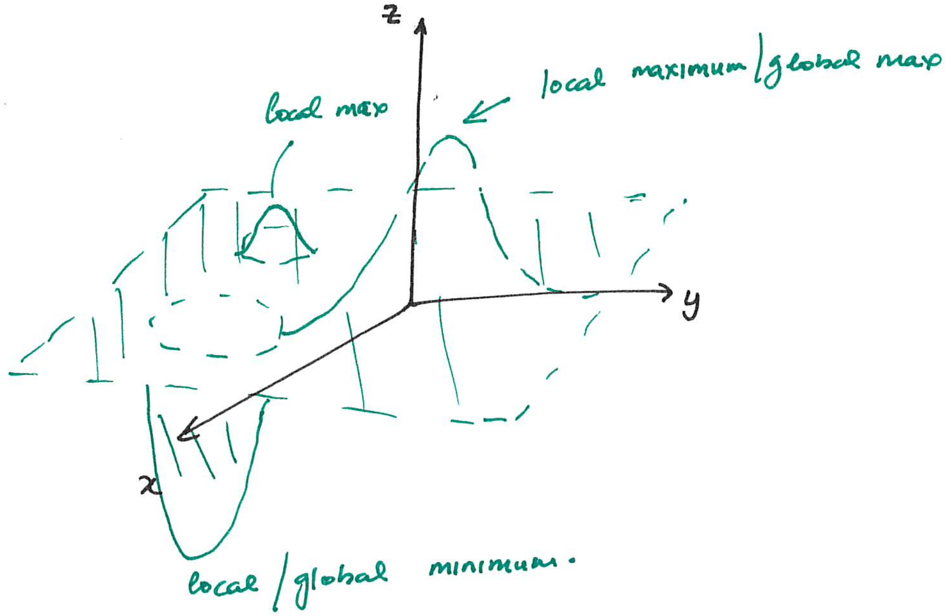
14.7 #1

Maxima and minima

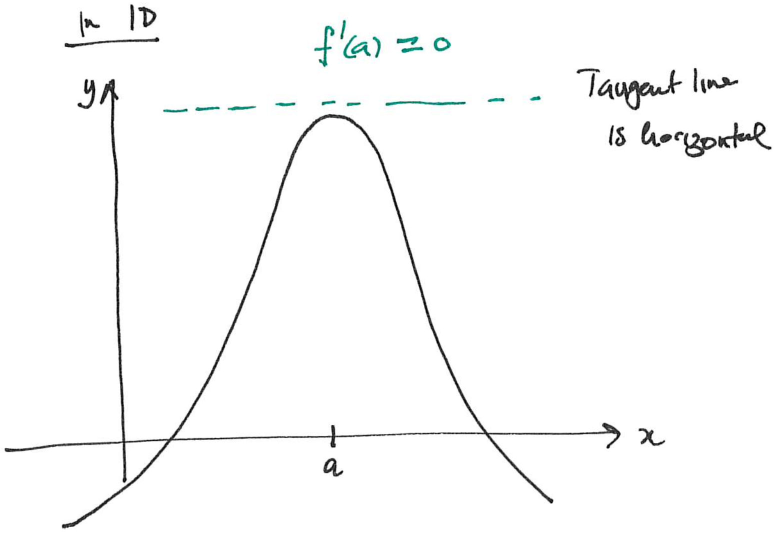


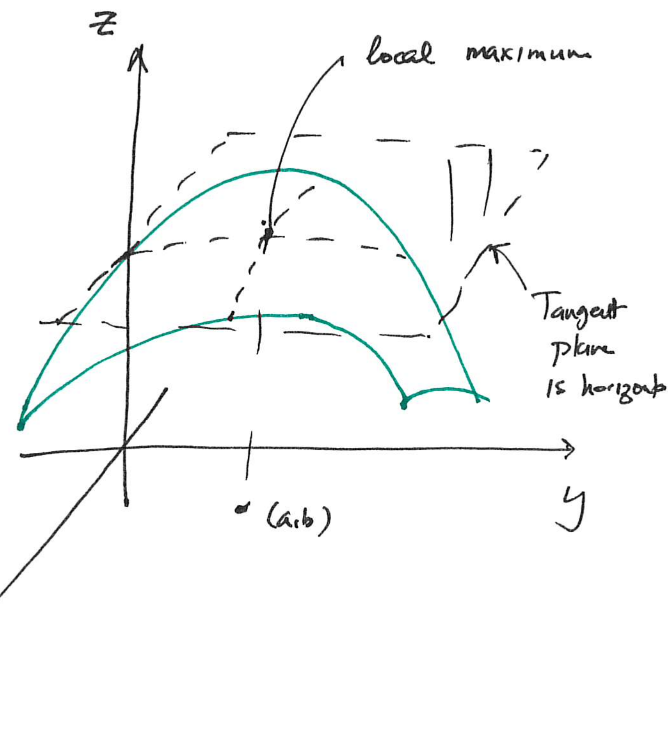
Local extreme values

- Local minimum : $f(x,y) \geq f(a,b)$ for all $(x,y) \in D(P,r)$
- Local maximum : $f(x,y) \leq f(a,b)$ for all $(x,y) \in D(P,r)$

CRITICAL POINT

In 1D





Critical point

A point $P = (a,b)$ in the domain of $f(x,y)$ is a critical point if

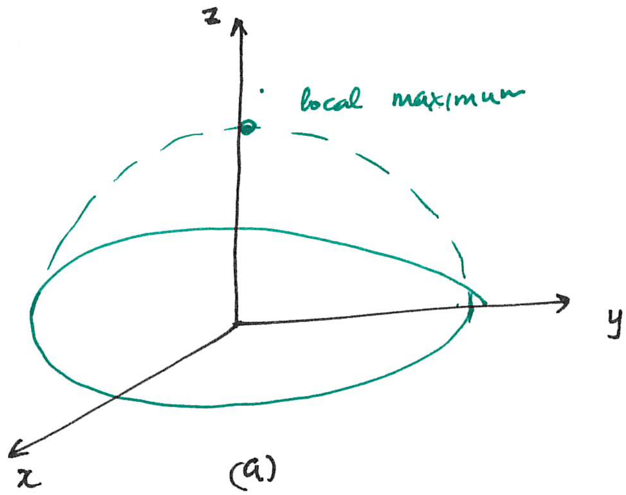
$f_x(a,b) = 0$ and $f_y(a,b) = 0$ OR

at least one of the partial derivatives does not exist

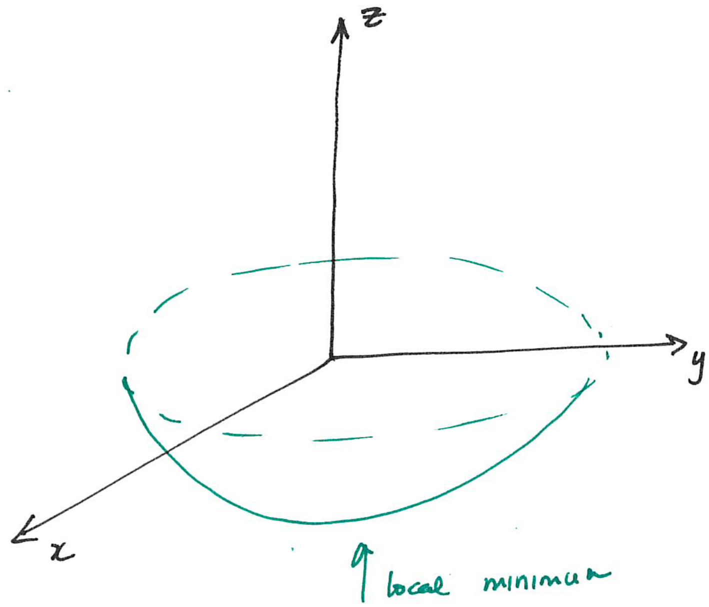
FACT

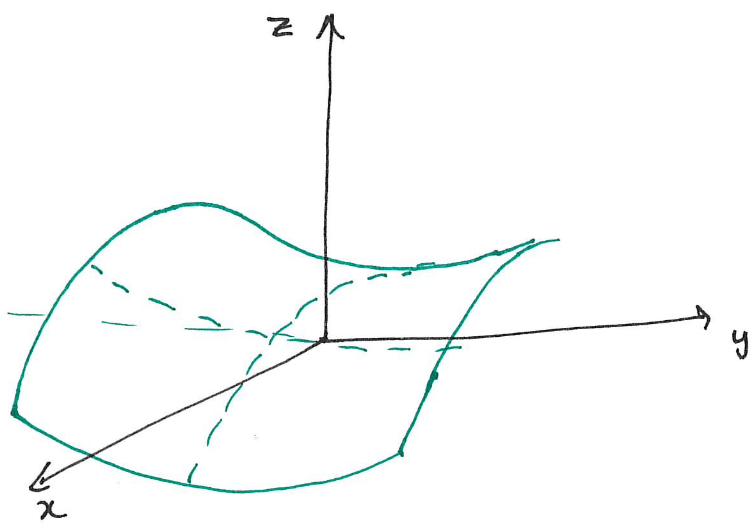
If $f(x,y)$ has a local minimum or maximum at $P=(a,b)$, then $P=(a,b)$ is a critical point.

Types of Critical points



(a)





Saddle.

Second derivative Test

In 1D, If $f''(c) \neq 0$ then $f''(c) < 0 \Rightarrow c$ is a local max

$f''(c) > 0 \Rightarrow c$ is a local min.

In 2D, we rely on the sign of the discriminant

$$D(a,b) = f_{xx}(a,b)\, f_{yy}(a,b) - \left(f_{xy}(a,b)\right)^2$$

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

FACT (SECOND DERIVATIVE TEST)

Let $P = (a,b)$ be a critical point of $f(x,y)$. Suppose the ~~continuous~~ partial derivatives are continuous near P, then

① If $D > 0$, $f(a,b)$ is a local min or local max

$f_{xx}(a,b) > 0 \Rightarrow f(a,b)$ is a local min

$f_{xx}(a,b) > 0 \Rightarrow f(a,b)$ is a local max.

② If $D < 0$, $f(a,b)$ is a saddle point.

③ If $D = 0$, the test is inconclusive.

Example

$f(x,y) = (x^2+y^2)e^{-x}$ → Find and classify the critical points of $f(x,y)$

$f_x(x,y) = -(x^2+y^2)e^{-x} + 2x e^{-x} = (2x - x^2 - y^2)\, e^{-x} = 0$

$f_y(x,y) = 2y\, e^{-x} = 0$

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$$(2x - x^2 - y^2)e^{-x} = 0 \quad \cdots (i)$$
$$2y e^{-x} = 0 \quad \cdots (ii)$$

From (ii) $y = 0$

Plug into (i)

$$2x - x^2 = 0 \Rightarrow x(2-x) = 0$$
$$x = 0$$
$$x = 2$$

Critical points are $(0,0)$ and $(2,0)$.

2nd order partials

$$f_{xx}(x,y) = \frac{\partial}{\partial x}(2x - x^2 - y^2)e^{-x} = +(2-2x)e^{-x} - (2x - x^2 - y^2)e^{-x}$$
$$= (2 - 4x + x^2 + y^2)e^{-x}$$

$$f_{yy}(x,y) = \frac{\partial}{\partial y}(2y e^{-x}) = 2e^{-x}.$$

$$f_{xy}(x,y) = f_{yx}(x,y) = \frac{\partial}{\partial x}(2ye^{-x}) = -2ye^{-x}$$ [partial derivatives are continuous]

Critical point	f_{xx}	f_{yy}	f_{xy}	D ($f_{xx}f_{yy} - (f_{xy})^2$)	Type
$(0,0)$	2	2	0	$2(2) - 0^2 = 4$	local max $D > 0$, $f_{xx} > 0$
$(2,0)$	$-2e^{-2}$	$2e^{-2}$	0	$-2e^{-2}(2e^{-2}) - 0^2$ $= -4e^{-4}$	Saddle point since $D < 0$.

Find 3 positive numbers whose sum is 100 and whose product is maximum

Let x, y and z be positive numbers

$x+y+z=100 \Rightarrow z = 100-x-y.$

We want to maximize $xyz = xy(100-x-y)$

$= 100xy - x^2y - xy^2 = f(x,y)$, $0<x,y,z<100$

$f_x = 100y - 2xy - y^2$

$f_y = 100x - x^2 - 2xy$

$f_x = 0 \Rightarrow 100y - 2xy - y^2 = 0$

$y(100-2x-y) = 0$

since $y>0$, $y = 100-2x$

$f_y = 0$ (plug in $y=100-2x$)

$100x - x^2 - 2x(100-2x) = 0$

$100x - x^2 - 200x + 4x^2 = 0$

$3x^2 - 100x = 0$

$x(3x-100) = 0$ $\quad x = \frac{100}{3}$ $(x>0)$

$y = 100 - 2\left(\frac{100}{3}\right)$

$\frac{300}{3} - \frac{200}{3} = \frac{100}{3}$

$f_{xx} = -2y$

$f_{yy} = -2x$

$f_{xy} = 100-2x-2y$

The only critical point is $\left(\frac{100}{3}, \frac{100}{3}\right)$

$D\left(\frac{100}{3}, \frac{100}{3}\right) = \left(\frac{-200}{3}\right)\left(-\frac{200}{3}\right) - \left(-\frac{100}{3}\right)^2 = \frac{10,000}{3} > 0$

$f_{xx}\left(\frac{100}{3}, \frac{100}{3}\right) = \frac{-200}{3} < 0$ so $f\left(\frac{100}{3}, \frac{100}{3}\right)$ is a local max!

$x=y=z=\frac{100}{3}$

Find the dimensions of the box w/ volume $1000 cm^3$ that has minimal surface area!

$V = xyz = 1000$

Surface area $= 2xy + 2xz + 2yz$.

We want to minimize $f(x,y) = 2xy + 2x\left(\frac{1000}{xy}\right) + 2y\left(\frac{1000}{xy}\right)$

$= 2xy + \frac{2000}{y} + \frac{2000}{x}$.

....

$x = 10$

$y = \frac{1000}{10^2} = 10$, $z = \frac{1000}{10^2} = 10$