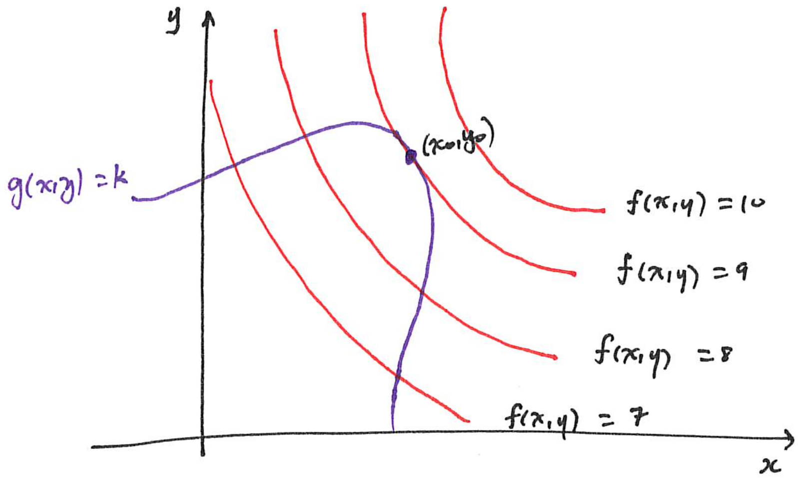
14.8 Lagrange Multipliers (Optimization) #1

Problem - Maximize or minimize $f(x,y,z)$ subject to a constraint $g(x,y,z)=k$.



SOLUTION

- Find a value c, such that the level curve $f(x,y)=c$ intersects $g(x,y)=k$
- This will occur when the curves just touch. ⇒ They will have a common tangent line!

The normal lines at (x_0,y_0) are ~~perpendicular~~ parallel so

$$\nabla f(x_0,y_0) = \lambda \nabla g(x_0,y_0) \quad \text{for some } \lambda \text{ (scalar)}$$

The same is true for maximize or minimize $f(x,y,z)$ subject to $g(x,y,z)=k$.

level surface of $f(x,y,z)=c$ is tangent to level surface of $g(x,y,z)=k$ so

$$\nabla f(x_0,y_0,z_0) = \lambda \nabla g(x_0,y_0,z_0).$$

Lagrange Multipliers

To maximize or minimize $f(x,y,z)$ subject to $g(x,y,z)=k$ (assuming $\nabla g \neq 0$ & extreme values exist)

(a) Find all values of x,y,z and λ such that

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$$
$$g(x,y,z) = k$$

(b) Evaluate f @ all points from (a). The largest value from that set is the max, smallest is min.

Find the maximum and minimum values of

$$f(x,y) = x^2+y^2 \text{ subject to } xy=1.$$

We want to find x, y, λ subject to $\nabla f(x,y) = \lambda g(x,y)$

$\nabla f(x,y) = \langle f_x, f_y \rangle = \langle 2x, 2y \rangle$

$\nabla g(x,y) = \langle y, x \rangle$

$\nabla f(x,y) = \lambda \nabla g(x,y)$

$2x = \lambda y \quad \dots (i)$

$2y = \lambda x \quad \dots (ii)$

$xy = 1 \quad \dots (iii)$

$y = \frac{1}{x}$ from (iii)

$2x = \lambda\left(\frac{1}{x}\right)$, from (ii) $\lambda = \frac{2y}{x} = \frac{2}{x^2}$

$2x = \left(\frac{2}{x^2}\right)\left(\frac{1}{x}\right) \Rightarrow 2x = \frac{2}{x^3}$

$2x^4 = 2$

$x^4 = 1 \Rightarrow x = \pm 1. \Rightarrow y = \frac{1}{x}$

$x = 1 \Rightarrow y = 1 \Rightarrow (1,1)$

$x = -1 \Rightarrow y = -1 \Rightarrow (-1,-1)$

$\lambda = \frac{2}{x^2} = \boxed{2}$

Evaluate f @ $(1,1)$ and $(-1,-1)$

$f(x,y) = x^2+y^2 = 1^2+(1)^2 = 2$

~~Maximum~~ Minimum value $= \boxed{2} = f(1,1) = f(-1,-1)$

$f(x,y) = x^2+y^2$

Since the constraint is $xy=1$,

x or y can be made arbitrarily large so there is no maximum.

Example #2

Find the extreme values of $f(x,y) = x^2 - y^2$ subject to the constraint $g(x,y) = x^2+y^2 = 1$.

$$\nabla f = \lambda \nabla g \Rightarrow \langle 2x, -2y \rangle = \lambda \langle 2x, 2y \rangle$$

So we have equations

$2x = 2\lambda x$ . . . (i)

$-2y = 2\lambda y$ . . . (ii)

$x^2+y^2=1$ . . . (iii)

from (i) $2x = 2\lambda x$

$2x(\lambda - 1) = 0 \Rightarrow x=0$ or $\lambda = 1$

If $x=0$, plug into (iii) yeilds $y^2=1 \Rightarrow y = \pm 1$

If $\lambda=1$, plug into (ii) $-2y = 2y \Rightarrow y=0$, from constraint $x = \pm 1$

So the possible points are $(0, \pm 1)$, $(\pm 1, 0)$

Evaluating f at these points $f(\pm 1, 0) = 1$

$f(0, \pm 1) = -1$