15.1

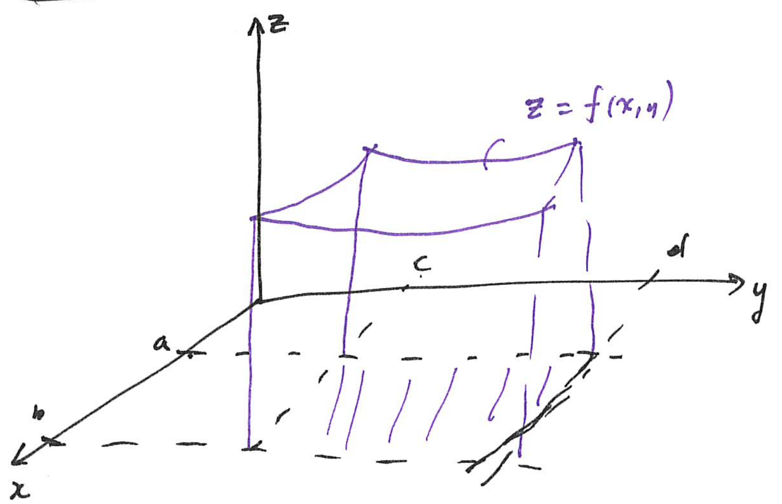
Multiple integrals

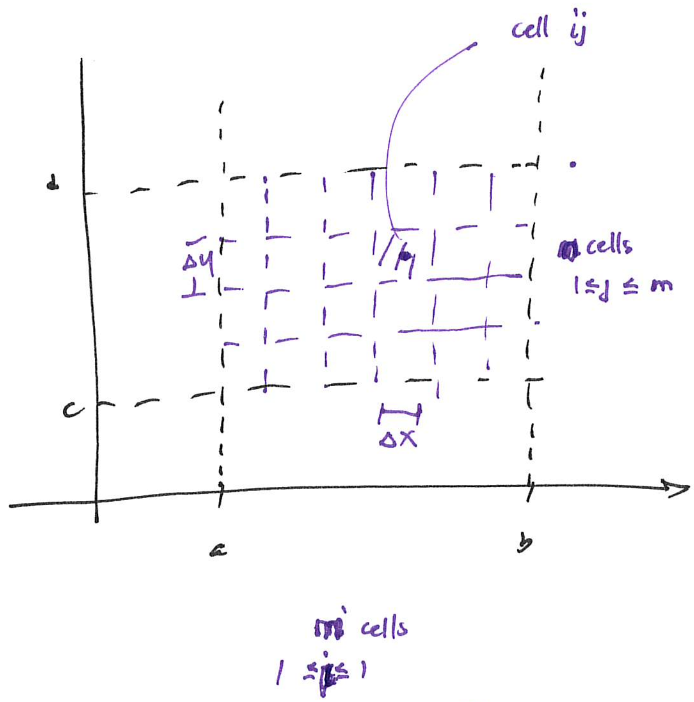
1. Recall the 1D definite integral (Area problem)

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\left(\sum_{i=1}^{n} f(x_i^*)\,\Delta x\right)$$

If $f(x)\geq 0$, the definite integral is the area under the curve.

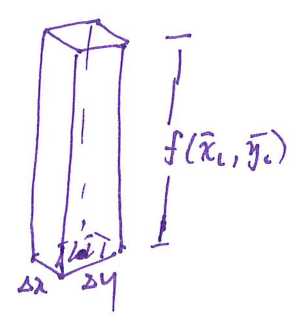
2. Volume Problem (Find the volume under $z=f(x,y)$ under $R=[a,b]\times[c,d]$.





1. For each cell pick the mid point $(\bar{x}_i,\bar{y}_i)$ and evaluate $f(\bar{x}_i,\bar{y}_i)$

2. Approximate the volume of cell ij



so that Volume $= \sum_{i=1}^{m}\sum_{j=1}^{n} f(\bar{x}_i,\bar{y}_i)\,\Delta x\,\Delta y$

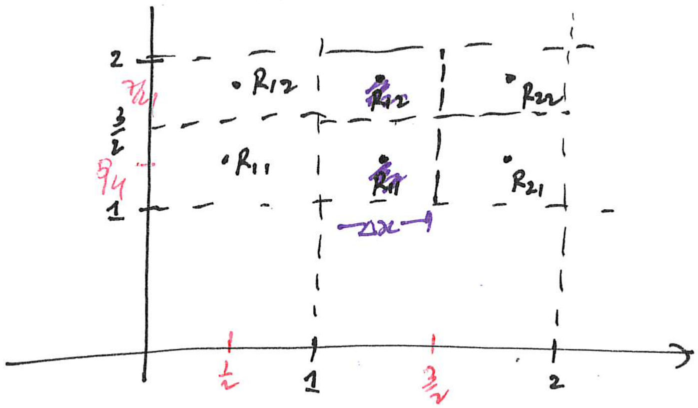
$= \sum_{i=1}^{m}\sum_{j=1}^{n} f(\bar{x}_i,\bar{y}_i)\,\Delta A$ [this is called a double Riemann integral].

$$= \iint_R f(x,y)\,dA$$

Example

Use the midpoint rule with $m=n=2$ to estimate $\iint_R (x-3y^2)\,dA$, where

$R = \{(x,y) \mid 0 \le x \le 2,\ 1 \le y \le 2\}$



$\Delta A = \frac{1}{\cancel{2}} \cdot \frac{1}{\cancel{2}} = 1$

$$\iint_R (x-3y^2)\,dA = \sum_{i=1}^{2}\sum_{j=1}^{2} f(\bar{x}_i, \bar{y}_j)\,\Delta A$$

$$= f(\bar{x}_1,\bar{y}_1)\,\Delta A + f(\bar{x}_1,\bar{y}_2)\,\Delta A + f(\bar{x}_2,\bar{y}_1)\,\Delta A + f(\bar{x}_2,\bar{y}_2)\,\Delta A$$

$$= f\left(\tfrac{1}{2},\tfrac{5}{4}\right)\Delta A + f\left(\tfrac{1}{2},\tfrac{7}{4}\right)\Delta A + f\left(\tfrac{3}{2},\tfrac{5}{4}\right)\Delta A + f\left(\tfrac{3}{2},\tfrac{7}{4}\right)\Delta A$$

$$= \cancel{-48}\ -\frac{95}{8} = -11.875$$

So $\iint_R (x-3y^2)\,dA = -11.875.$

Properties

① $\iint_R [f(x,y)+g(x,y)]\,dA = \iint_R f(x,y)\,dA + \iint_R g(x,y)\,dA$

② $\iint_R c f(x,y)\,dA = c\iint_R f(x,y)\,dA$. c is a constant

If $f(x,y) \ge g(x,y)$ in R, then

③ $\iint_R f(x,y)\,dA \ge \iint_R g(x,y)\,dA$

Average value of a function

$$f_{ave} = \frac{1}{b-a} \int_a^b f(x)\,dx \quad \text{In 1D},$$

In 3D

$$f_{ave} = \frac{1}{A(R)} \iint_R f(x,y)\,dA$$

Iterated integrals (from the FTC)!

Let $R = [a,b] \times [c,d]$.

We want to calculate $\iint_R f(x,y)\, dA$.

- $\int_c^d f(x,y)\, dy$ – x is fixed, integrate w.r.t y. [partial integral w.r.t y]
- $\int_c^d f(x,y)\, dy = A(x)$.
- We can now integrate $A(x)$ w.r.t x from $x=a$ to $x=b$

$$\int_a^b A(x)\,dx = \int_a^b \left[\int_c^d f(x,y)\, dy\right] dx$$

(This is called an iterated integral).

$$= \int_a^b \int_c^d f(x,y)\, dy\, dx$$

$\Rightarrow$ integrate w.r.t y first, then x

Similarly $\int_c^d \int_a^b f(x,y)\, dx\, dy = \int_c^d \left[\int_a^b f(x,y)\, dx\right] dy$ $\Rightarrow$ integrate w.r.t x first, then y.

Example #1

$$\int_1^4 \int_0^2 (6x^2y - 2x)\, dy\, dx$$

$$3x^2y^2 - 2xy \Big|_{y=0}^{y=2}$$

$$\int_0^2 (6x^2y - 2x)\, dy = \frac{6x^2y^2}{2} - 2xy \Big|_{y=0}^{y=2} = 3x^2(4) - 2x(2)$$

$$= 12x^2 - 4x$$

$$2(2x^3 - x^2)\Big|_1^4$$

$$\int_1^4 \int_0^2 (6x^2y - 2x)\, dy\, dx = \int_1^4 (12x^2 - 4x)\, dx$$

$$= \boxed{\frac{48x^3}{3} - \frac{4x^2}{2}\Big|_{x=1}^{4}}$$ stop $= 4x^3 - 2x^2\Big|_{x=1}^{4} = 246$

~~248~~ –

~~492 – (19)~~

978

$$= \left[\frac{48}{3}(4)^3 - 2(4)^2\right] - \left[\frac{48}{3}(1)^3 - 2(1)^2\right] = c$$

15.2

Average value of a function.

$A = [a,b] \times [c,d]$

In 1D

$$f_{ave} = \frac{1}{b-a} \int_a^b f(x)\,dx$$

, In 2D

$$f_{ave} = \frac{1}{|A|} \int_c^d \int_a^b f(x,y)\,dx\,dy.$$

Fubini's Theorem (In general)

If f is continuous on $R = \{(x,y) \mid a \le x \le b,\ c \le y \le d\}$

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

Special case

If $f(x,y)$ can be factored into $g(x)\,h(y)$ and $R = [a,b] \times [c,d]$

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b g(x)\,h(y)\,dx\,dy$$

$$= \int_c^d \left[\int_a^b g(x)\,h(y)\,dx \right] dy$$

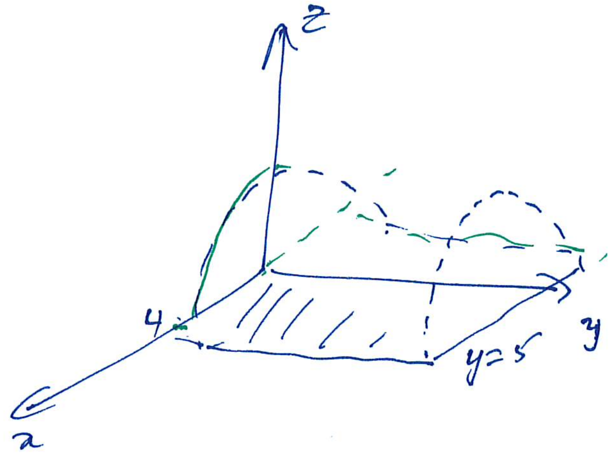
Notice that in the inner integral $h(y)$ is a constant, therefore

$$= \int_c^d \left[h(y) \left(\int_a^b g(x)\,dx \right) \right] dy$$

$$= \int_a^b g(x)\,dx \int_c^d h(y)\,dy$$

So in general

$$\iint_R g(x)\,h(y)\,dA = \int_a^b g(x)\,dx \int_c^d h(y)\,dy, \quad R = [a,b] \times [c,d].$$



$$V = \int_0^5 \int_0^4 (16 - x^2)\, dx\, dy = \int_0^4 (16 - x^2)\, dx \int_0^5 dy$$

$$= 16x - \frac{x^3}{3} \Big|_0^4 \ [y]_0^5$$

$$= \frac{640}{3}$$

Example

$$\iint_R \frac{xy^2}{x^2+1}\, dA \qquad R = \{(x,y) \mid 0 \le x \le 1,\ -3 \le y \le 3\}$$

$$= \int_0^1 \int_{-3}^3 \frac{xy^2}{x^2+1}\, dy\, dx = \left(\int_0^1 \frac{x}{x^2+1}\, dx\right)\left(\int_{-3}^3 y^2\, dy\right)$$

$$\left(\frac{x}{x^2+1}\right) y^2$$

$$= \frac{1}{2} \ln|x^2+1| \Big|_0^1 \left[\frac{1}{3} y^3\right]_{-3}^3$$

$$= \frac{1}{2}(\ln 2 - \ln 1) \cdot \left(\frac{1}{3}(27+27)\right)$$

$$= \underline{9 \ln 2}$$

$$\int \frac{x}{x^2+1}\, dx$$

$u = x^2+1$

$\frac{du}{dx} = 2x$

$\frac{du}{2} = x\,dx$

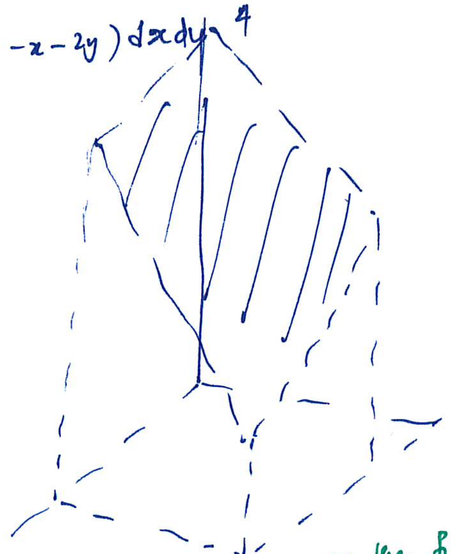
$\frac{1}{2}\int \frac{1}{u}$

$= \frac{1}{2}\ln|u| + C$

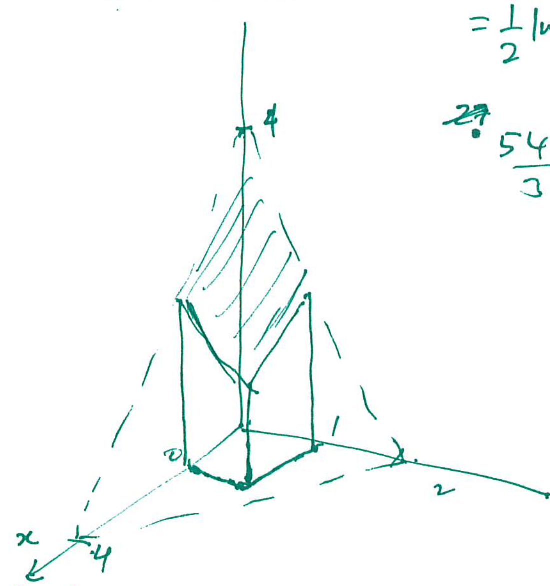
$\frac{54}{3 \cdot 2} \quad \frac{54}{6} = 9$

Sketch the region

$$\int_0^1 \int_0^1 (4 - x - 2y)\, dx\, dy$$



$z = 4 - x - 2y$



Set up

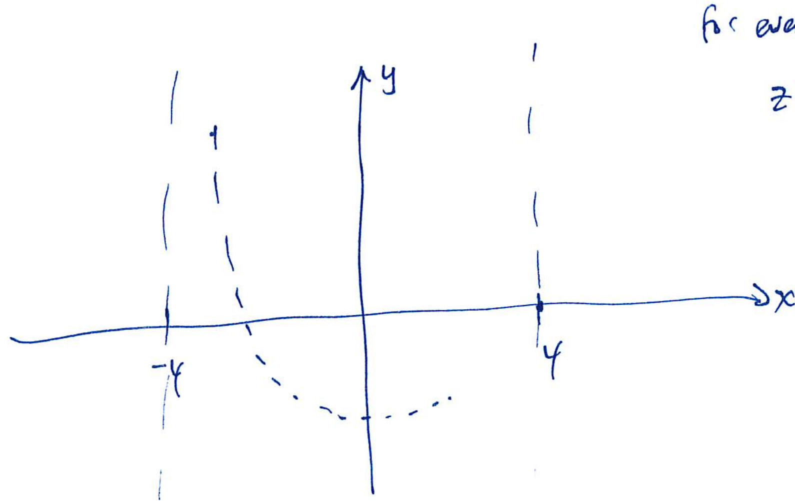
Find the volume of the solid in the first octant enclosed by $\underline{z = 16 - x^2}$ and the plane $y = 5$

$z = 16 - x^2$

- level curves

$z = 0 \quad x^2 = 16$

$x = \pm 4$



for every value of y

$z = 16 - x^2$