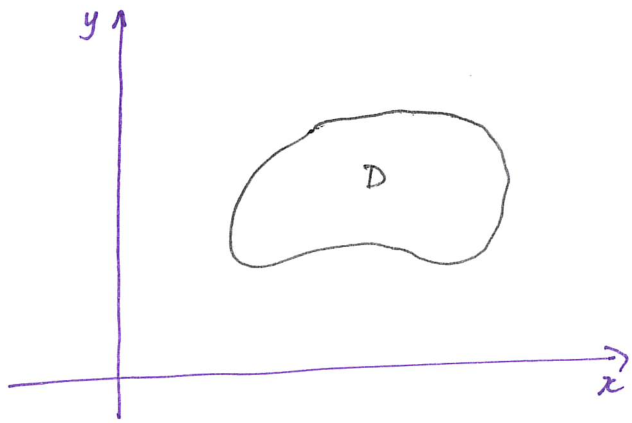
Double integrals over general Regions

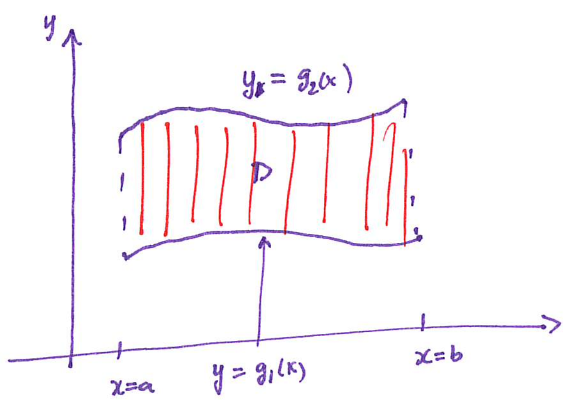
Goal

Compute $\iint_D f(x,y)\, dA$, where D is a general region in 2D



Types of Regions.

I VERTICALY ORIENTED



$$\boxed{D = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}}.$$

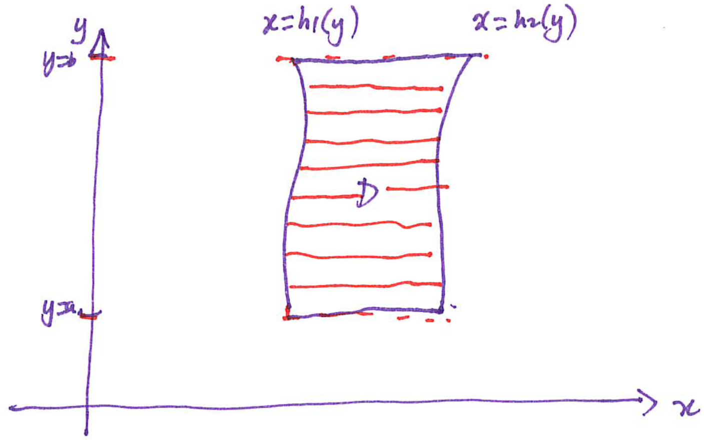
$$\iint_D f(x,y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx$$

x limits   y limits.

Notes

- We have to perform the integral in y first.

II HORIZONTALY ORIENTED



$$D = \{(x,y) \mid c \le y \le d \ , \ h_1(y) \le x \le h_2(y)\}$$

$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

Example #1 Evaluate $\iint_D (1+2y)\,dA$ where $D = \{(x,y) \mid 0 \le x \le 1, \ x^2 \le y \le x\}$

D is the region bounded by $y = x^2$ and $y = x$.

$$\int_0^1 \int_{x^2}^{x} (1+2y)\,dy\,dx$$

Inner integral

$$\int_{x^2}^{x} (1+2y)\,dy = y + \frac{2y^2}{2}\Big|_{y=x^2}^{x} = y + y^2\Big|_{y=x^2}^{x}$$

$$= [x + x^2] - [x^2 + x^4]$$

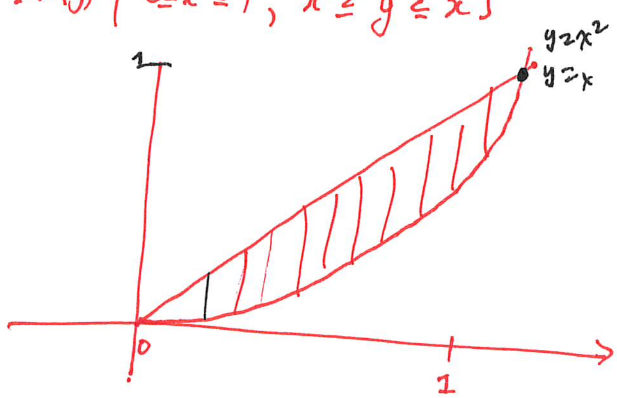
$$= x - x^4$$

$$\int_0^1 \int_{x^2}^{x} (1+2y)\,dy\,dx = \int_0^1 (x - x^4)\,dx = \frac{x^2}{2} - \frac{x^5}{5}\Big|_{x=0}^{1}$$

$$= \frac{1}{2} - \frac{1}{5} = \frac{3}{10}$$

Region of integration

$$D = \{(x,y) \mid 0 \le x \le 1 \ , \ x^2 \le y \le x\}$$



D is a vertically oriented region

Horizontally

$0 \le y \le 1$

$y \le x \le \sqrt{y}$

$$\int_0^1 \int_y^{\sqrt{y}} (1+2y)\,dx\,dy$$

$$x + 2yx\Big|_{x=y}^{\sqrt{y}}$$

Example #2

Set up $\iint_D x^3\,dA$, $D = \{(x,y) \mid 1 \le x \le e,\ 0 \le y \le \ln(x)\}$

$$\iint_D x^3\,dA = \int_1^e \int_0^{\ln(x)} x^3\,dy\,dx.$$

Evaluate ex

$\qquad\quad v \qquad u$

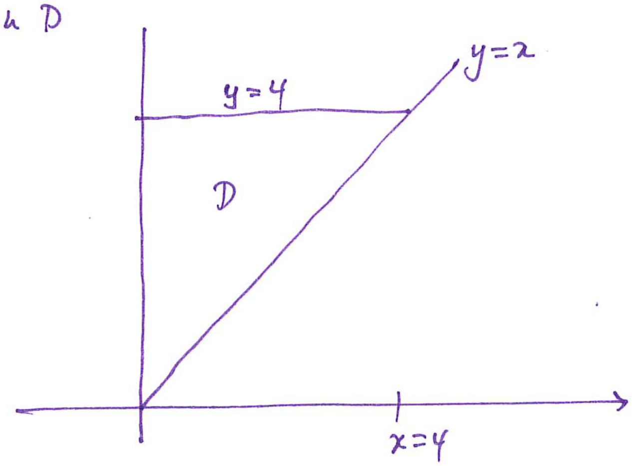
$x^3 \ln(x)$

$uv - \int v\,du$

Example #3

$\iint_D y^2 e^{xy}\,dA$, D is the region bounded by $y=x$, $y=4$, $x=0$.

Sketch D



We have a choice here, D can be described as both a horizontal and vertical region.

Which is best?

Type 1. (Vertical)

$$\boxed{\begin{array}{l} y=x \rightarrow y=4 \\ x=0 \rightarrow x=4 \end{array}}$$

Start with the x variable.

$$\iint_D y^2 e^{xy}\,dA = \int_0^4 \int_x^4 y^2 e^{xy}\,dy\,dx \qquad I1$$

Which one is the easier one?

Type II (Horizontal)

$0 \le y \le 4$

$0 \le x \le y$

$$\iint_D y^2 e^{xy}\,dA = \int_0^4 \int_0^y y^2 e^{xy}\,dx\,dy \qquad I2$$

I1

Inner integral first

$$\int_x^4 y^2 e^{xy}\,dy$$

Integrate by parts

let $u = y^2$ $\quad dv = e^{xy}dx$

$\frac{du}{dy} = 2y$ $\quad v = \frac{1}{x}e^{xy}$

$$\int y^2 e^{xy}\,dy = \frac{y^2}{x}e^{xy} - \boxed{\int \frac{1}{x}e^{xy}\,2y\,dy}$$

Integrate by parts again!

$u = 2y$ $\quad dv = \frac{1}{x}e^{xy}dy$

$\frac{du}{dy} = 2$ $\quad v = \frac{1}{x^2}e^{xy}$

$$\int \frac{1}{x}e^{xy}2y\,dy = \frac{2y}{x^2}e^{xy} - \int \frac{1}{x^2}e^{xy}\,2\,dy$$

$$= \frac{2}{x^2}e^{xy} - \frac{1}{x^3}e^{xy}.$$

so

$$\int_x^4 y^2 e^{xy} = \frac{y^2}{x}e^{xy} - \left[\frac{2}{x^2}e^{xy} - \frac{1}{x^3}e^{xy}\right]\Bigg|_x^4$$

$\cancel{\int_0^4\int_x^4 y^2 e^{xy}} \Rightarrow \int$

⋮ plug in, and keep going!

Requires integration by parts

I2

$$\int_0^4\int_0^y y^2 e^{xy}\,dx\,dy$$

Inner integral

$$\int_0^y y^2 e^{xy}\,dx$$

let $u = xy$

$\frac{du}{dx} = y$

$\frac{du}{y} = dx$

$$= \frac{y^2}{y}e^{xy} = ye^{xy}\Big|_{x=0}^{y}$$

$$= ye^{y^2} - ye^0 = ye^{y^2} - y$$

$$\int_0^4\int_0^y y^2 e^{xy}\,dx\,dy = \int_0^4 \left(ye^{y^2} - y\right)dy$$

let $u = y^2$

$\frac{du}{dy} = 2y$

$\frac{du}{2} = y\,dy$

$$\int ye^{y^2}dy = \frac{1}{2}\int e^u\,du$$

$$= \frac{1}{2}e^u$$

$$= \frac{1}{2}e^{y^2}.$$

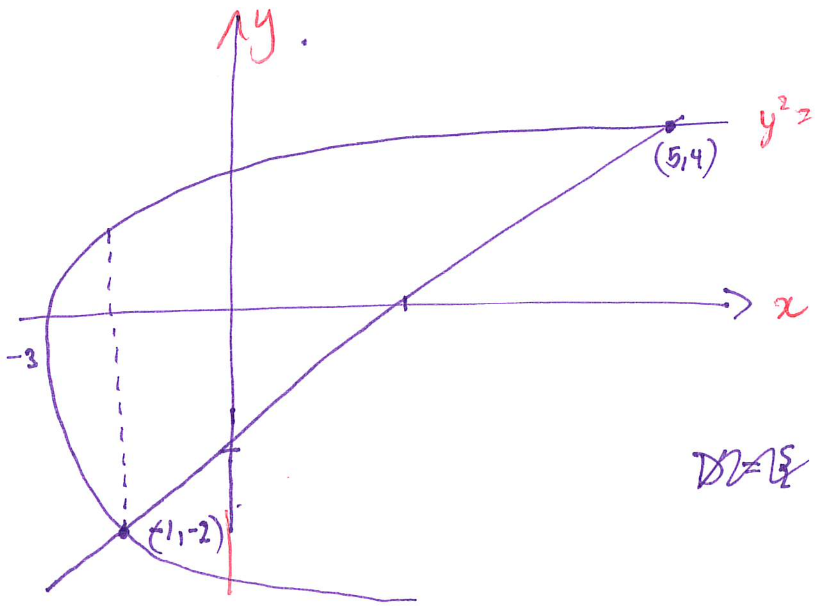
$$\int_0^4 ye^{y^2} - y = \frac{1}{2}e^{y^2} - \frac{y^2}{2}\Big|_0^4$$

$$= \left(\frac{1}{2}e^{16} - 8\right) - \left(\frac{1}{2}\right)$$

$$\int_0^4\int_0^y y^2 e^{xy}\,dx\,dy = \frac{e^{16}}{2} - \frac{17}{2}.$$

Type I Regions vs Type II

~~Find~~ Evaluate $\iint_D xy\,dA$, where D is the region bounded by $y = x-1$ and the parabola $y^2 = 2x+6$.



$(x-1)^2 = 2x+6$

$x^2-2x+1 = 2x+6$

$x^2-4x-5=0$

$x^2+x-5x-5=0$

$x(x+1)-5(x+1)=0$

$x=-1,\ x=5$

~~D = {~~

As Type I (Vertical)

$$\iint_D xy\,dA = \int_{-3}^{-1}\int_{-\sqrt{2x+6}}^{\sqrt{2x+6}} xy\,dy\,dx + \int_{-1}^{5}\int_{x-1}^{\sqrt{2x+6}} xy\,dy\,dx$$

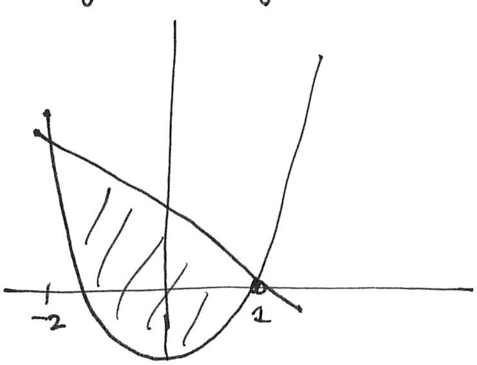
As Type II

$D = \{(x,y) \mid -2 \le y \le 4,\ \tfrac{1}{2}y^2-3 \le x \le y+1\}$

$$\iint_D xy\,dA = \int_{-2}^{4}\int_{\frac{1}{2}y^2-3}^{y+1} xy\,dx\,dy$$, much better!

Example

Set up the integral for the volume under the plane $x-2y+z=1$ above the region $x+y=1$ and $y=x^2-1$



$$\int_{-2}^{1}\int_{x^2-1}^{1-x} (1-x+2y)\,dy\,dx$$