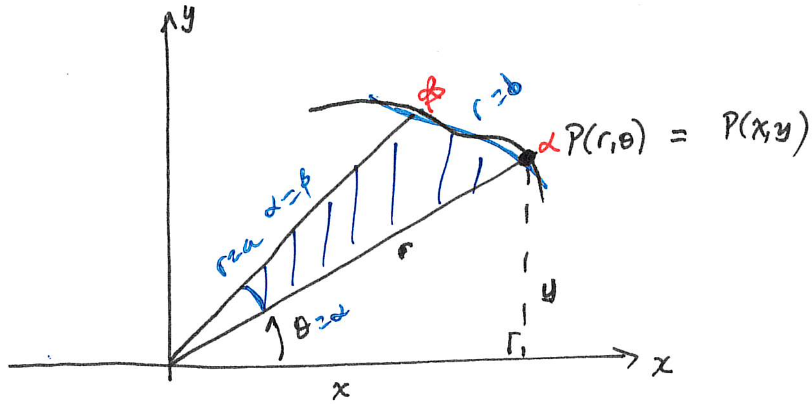
15.3 Double Integrals in Polar Coordinates.

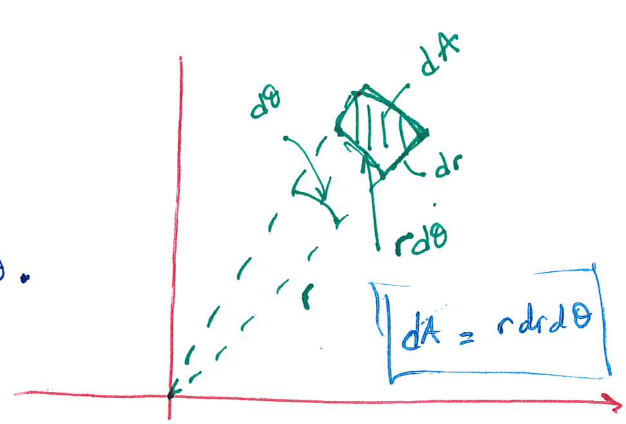


$r^2 = x^2 + y^2$

$x = r\cos\theta$

$y = r\sin\theta$.

Infintesimal polar Rectangle.



$$\boxed{dA = r\,dr\,d\theta}$$

$$R = \{(r,\theta) \mid a \le r \le b,\ \alpha \le \theta \le \beta\}.$$

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta.$$

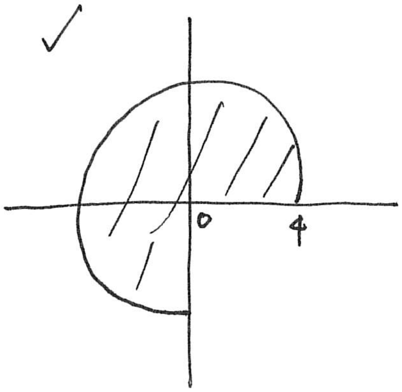
Example #1

Types of Regions

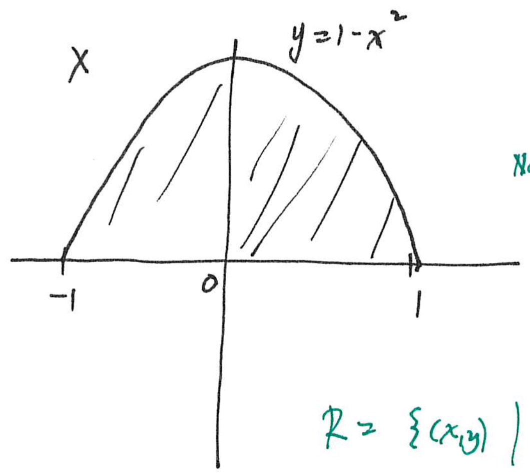
$$C = 2\pi r \frac{d\theta}{2\pi} = r\,d\theta$$

$$= r\,d\theta$$

✓



X



No polar, rectangular works better

$$R = \{(x,y) \mid -1 \le x \le 1,\ 0 \le y \le 1 - x^2\}$$

$$R = \{(r,\theta) \mid 0 \le r \le 4,\ 0 \le \theta \le \tfrac{3\pi}{2}\}$$

~~$R = \{(r,\theta) \mid \ldots\}$~~

Rectangular coordinates are easier

Example #2

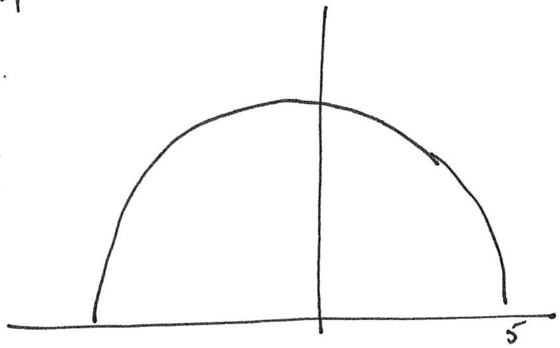
$$\iint_D x^2 y\,dA$$, D = top half of disk with center @ (0,0) and r = 5

18.4

#2.

$D = \{(r,\theta) \mid 0 \le r \le 5,\ 0 \le \theta \le \pi\}$

$x = r\cos\theta \qquad y = r\sin\theta$



$$\iint_D x^2 y\, dA = \int_0^{\pi}\int_0^5 (r\cos\theta)^2\, r\sin\theta\ r\,dr\,d\theta$$

$$= \int_0^{\pi}\int_0^5 r^3\cos^2\theta\,\sin\theta \cdot r\,dr\,d\theta = \left(\int_0^5 r^4\,d\theta\right)\left(\int_0^{\pi}\cos^2\theta\sin\theta\,d\theta\right)$$

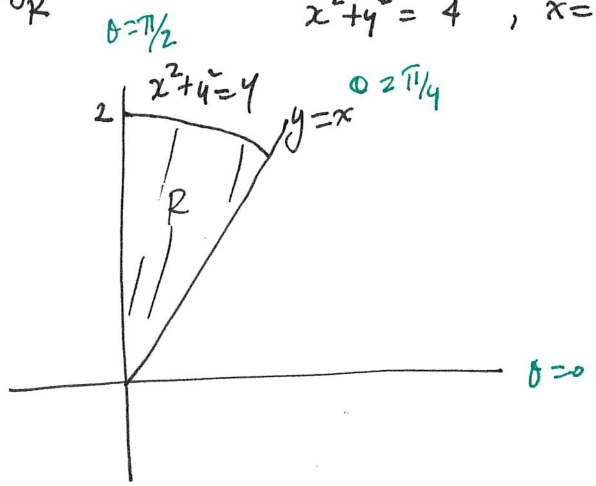
Recall

$$= \left(\frac{r^5}{5}\Big|_{r=0}^{5}\right)\left(-\frac{1}{3}\cos^3(\theta)\Big|_{\theta=0}^{\pi}\right)$$

$$-\frac{1}{3}(-1-1)\cdot 625 = \frac{1250}{3}.$$

Set up

$\iint_R (2x-y)\,dA$, where R is the region in the first quadrant enclosed by $x^2+y^2=4$, $x=0$ and $y=x$

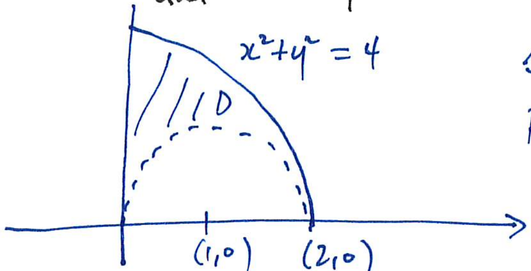


$R = \{(r,\theta) \mid 0 \le r \le 2,\ \pi/4 < \theta < \frac{\pi}{2}\}$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^2 (2r\cos\theta - r\sin\theta)\,r\,dr\,d\theta \qquad \text{ANS} \quad \frac{16}{3} - 4\sqrt{2}.$$

$$= \left(\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(2\cos\theta - \sin\theta)\,d\theta\right)\left(\int_0^2 r^2\,dr\right)$$

Set up $\iint_D x\,dA$, where D is the region in the first quadrant between $x^2+y^2=4$ and $x^2+y^2=2x$



$\int_0$

$R = \{(r,\theta) \mid 2\cos\theta \le r \le 2\},\ 0 \le \theta \le \frac{\pi}{2}\}$

$x^2+y^2 = 2x$

Complete the square in x

$x^2 - 2x + 1 - 1 + y^2 = 0$

$(x-1)^2 + y^2 = 1$

$r^2\cos^2\theta + r^2\sin^2\theta = 2r\cos\theta$

$r^2 = 2r\cos\theta$

$$\iint_D x\,dA = \int_0^{\frac{\pi}{2}} \int_{2\cos\theta}^{2} r\cos\theta \; r\,dr\,d\theta$$

Inner integral $\Rightarrow$ $\displaystyle\int_{2\cos\theta}^{2} r^2 \cos\theta \, dr$

$$= \frac{r^3}{3}\cos\theta \Big|_{r=2\cos\theta}^{2} = \frac{8}{3}\cos\theta - \frac{(2\cos\theta)^3}{3}\cos\theta$$

$$= \frac{8}{3}\cos\theta - \frac{8}{3}\cos^4\theta$$

$$\int_0^{\frac{\pi}{2}} \int_{2\cos\theta}^{2} r\cos\theta \; r\,dr\,d\theta = \frac{8}{3}\int_0^{\frac{\pi}{2}} \left(\cos\theta - \cos^4\theta\right) d\theta$$

Recall that

$$\cos^4\theta = \cos^2\theta \cdot \cos^2\theta$$

$$= \frac{1}{2}\left(1+\cos 2\theta\right)^2 = \frac{1}{4}\left(1 + 2\cos 2\theta + \cos^2 2\theta\right)$$

$$= \frac{1}{4}\left(1 + 2\cos 2\theta + \frac{1}{2}\left(1 + \cos 4\theta\right)\right)$$

$$= \frac{1}{4}\left(\frac{3}{2} + 2\cos 2\theta + \frac{1}{2}\cos 4\theta\right)$$

$$\frac{8}{3}\int_0^{\frac{\pi}{2}} \cos\theta - \cos^4\theta \, d\theta = \frac{8}{3}\left[\sin\theta - \frac{1}{4}\left\{\frac{3}{2}\theta + \sin 2\theta + \frac{1}{8}\sin 4\theta\right\}\right]_{\theta=0}^{\pi/2}$$

$$= \frac{8}{3}\left[1 - \frac{1}{4}\left(\frac{3}{2}\cdot\frac{\pi}{2}\right)\right]$$

$$= \frac{8}{3}\left[1 - \frac{3\pi}{16}\right] = \frac{16-3\pi}{16}\cdot\frac{8}{3} = \frac{16-3\pi}{6}$$