15.8 Triple Integrals

$$\iiint_B f(x,y,z)\, dV$$

GOAL – define an integral of a 3 variable function.

$$\iiint_B f(x,y,z)\, dV = \lim_{l,m,n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_i^*, y^*, z^*)\, \delta V.$$

Let $B = [a,b] \times [c,d] \times [r,s]$

$$\iiint_B f(x,y,z)\, dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\, dx\, dy\, dz$$

Example #1

Evaluate $\displaystyle\iiint_B (xy + z^2)\, dV$

$B = \{ (x,y,z) \mid 0 \le x \le 2, \quad 0 \le y \le 1, \quad 0 \le z \le 3 \}$ [Rectangular box]

$$\int_0^3 \int_0^1 \int_0^2 xy + z^2\, dx\, dy\, dz \quad = \underline{21}$$ Work your way outwards!

= start with

① $\displaystyle\int_0^2 xy + z^2\, dx = \frac{x^2}{2} y + z^2 x \Big|_0^2 = 2y + 2z^2$

$\displaystyle\int_0^1 2y + 2z^2\, dy = \frac{2y^2}{2} + \cancel{2z^3}\ 2z^2 y \Big|_{y=0}^{1}$

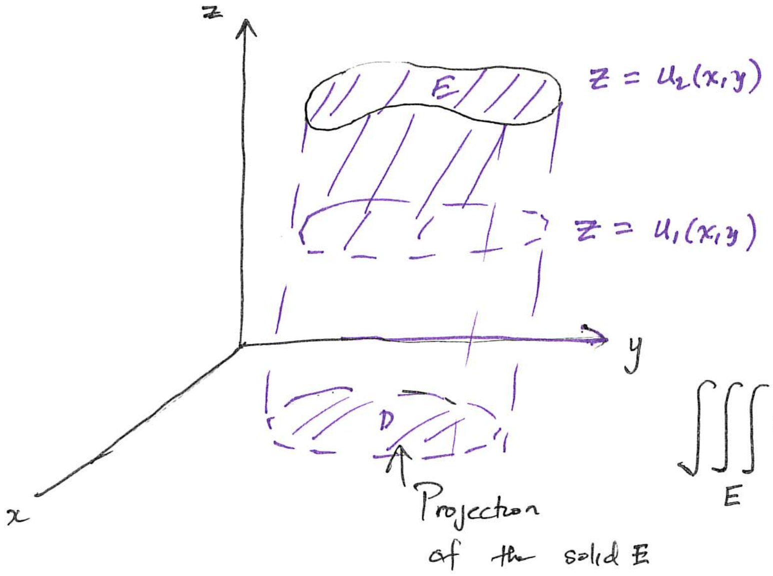
$\displaystyle\frac{2}{2} + 2z^2 = 1 + 2z^2$

$\displaystyle\int_0^3 1 + 2z^2\, dz = z + \frac{2z^3}{3}\Big|_0^3$

$\displaystyle 3 + \frac{2}{3}\cdot 3^3 = 3 + 18 = 21$

Triple integrals over general regions.

Type I (Regions of type E).

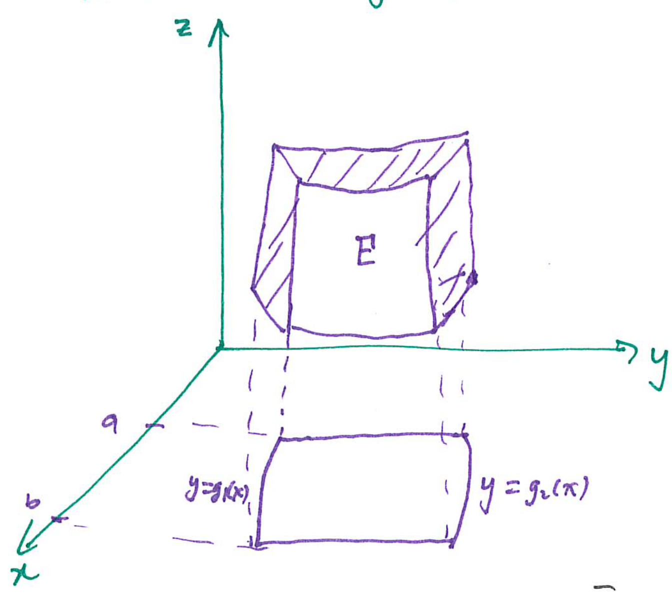


$$E = \{ (x,y,z) \mid (x,y) \in D,\ u_1(x,y) \le z \le u_2(x,y) \}$$

$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right] dA$$

$$\iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right] dA$$

For the outer integral, If D is vertically oriented.



$$E = \{ (x,y,z) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x),\ u_1(x,y) \le z \le u_2(x,y) \}$$

$$\iiint_E f(x,y,z)\,dV$$

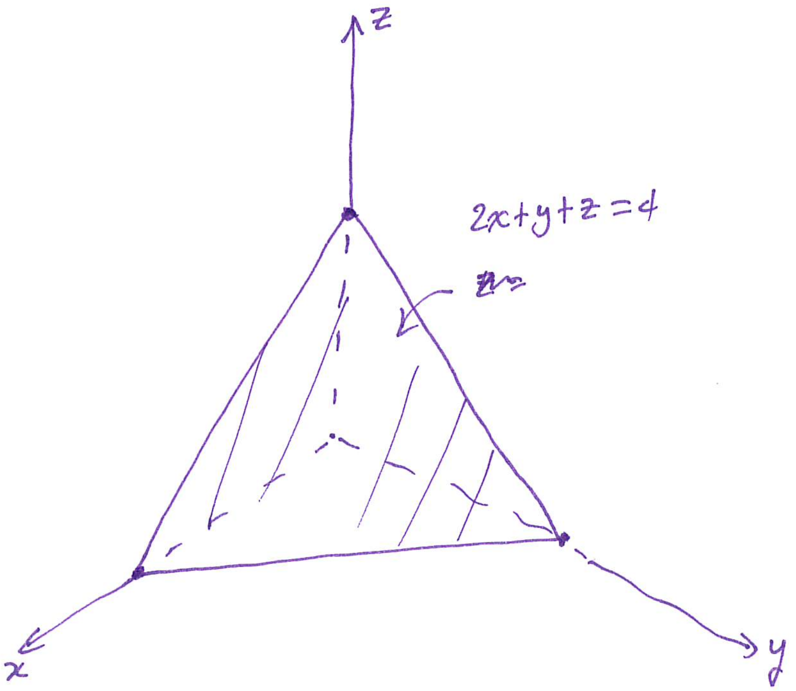
$$= \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dy\,dx.$$

If D is horizontally oriented

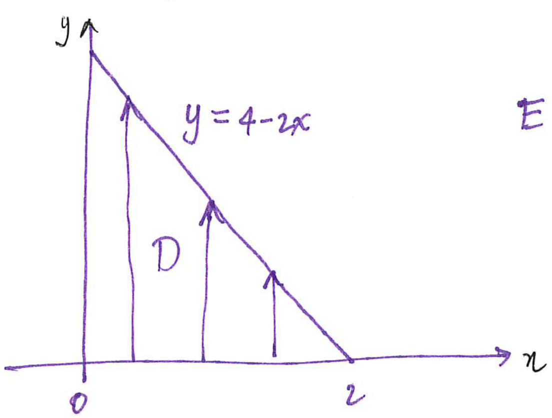
$$E = \{ (x,y,z) \mid c \le y \le d,\ h_1(y) \le x \le h_2(y),\ u_1(x,y) \le z \le u_2(x,y) \}$$

$$\iiint_E f(x,y,z)\,dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dx\,dy.$$

Evaluate $\iiint_E z\,dV$, where E is the solid tetrahedron bounded by the coordinate planes and $2x+y+z=4$.



$2x+y+z=4$ intersects $z=0$ along $2x+y=4 \Rightarrow y=4-2x$



$E = \{(x,y,z) \mid 0 \le x \le 2,\ 0 \le y \le 4-2x,\ 0 \le z \le 4-2x-y\}$

$$V = \int_0^2 \int_0^{4-2x} \int_0^{4-2x-y} z\,dz\,dy\,dx.$$

① $\displaystyle\int_0^{4-2x-y} z\,dz = \frac{z^2}{2}\Bigg|_0^{4-2x-y} = \frac{1}{2}(4-2x-y)^2$

② $\displaystyle\frac{1}{2}\int_0^{4-2x} (4-2x-y)^2\,dy \ \ldots$

$$\frac{1}{2}\int_0^{4-2x}(4-2x-y)^2\,dy.$$

$$\frac{1}{2}\int(4-2x-y)^2\,dy \xrightarrow[u=4-2x-y]{} -\frac{1}{2}\int u^2\,du = -\frac{1}{2}\frac{u^3}{3} = -\frac{1}{6}(u^3)$$

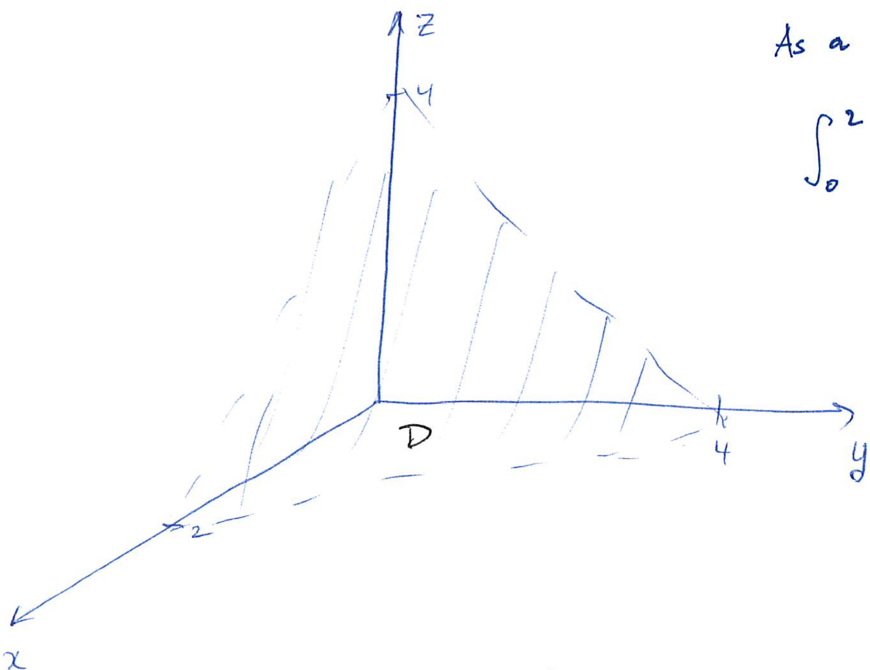
$$\frac{du}{dy} = -1$$

$$du = -dy$$

$$= -\frac{1}{6}(4-2x-y)^3\Big|_0^{4-2x}$$

Example #16

Find the volume of the tetrahedron enclosed by the coordinate planes and the plane $2x + y + z = 4$.

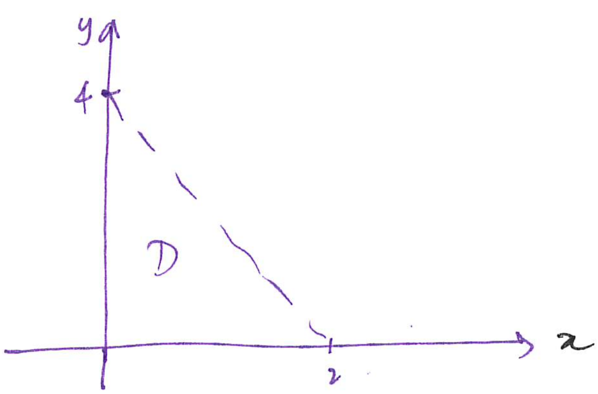


As a double integral: we know

$$\int_0^2 \int_0^{4-2x} 4 - 2x - y \, dy \, dx$$

but this is equal to a triple integral, which one!.

$2x + y + z = 4$ intersects the xy plane along $2x + y = 4 \Rightarrow y = 4 - 2x$



$$E = \{(x, y, z) \mid 0 \le x \le 2,\ 0 \le y \le 4 - 2x,\ 0 \le z \le 4 - 2x - y\}$$

$$V = \int_0^2 \int_0^{4-2x} \int_0^{4-2x-y} 1 \, dz \, dy \, dx$$

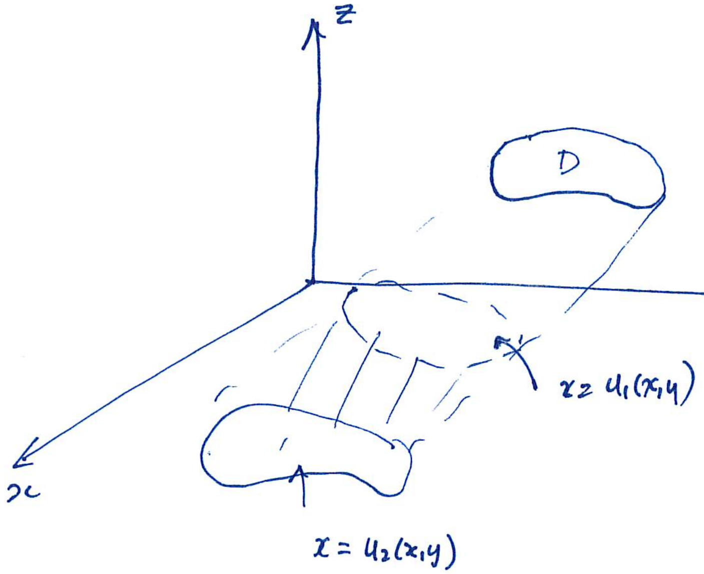
$$V = \int_0^2 \int_0^{4-2x} (4 - 2x - y) \, dy \, dz. \quad = \frac{16}{3}$$

$$\cancel{V = \int_0^2 \int_0^{4-}}$$

Regions of Type III

15.6 #5

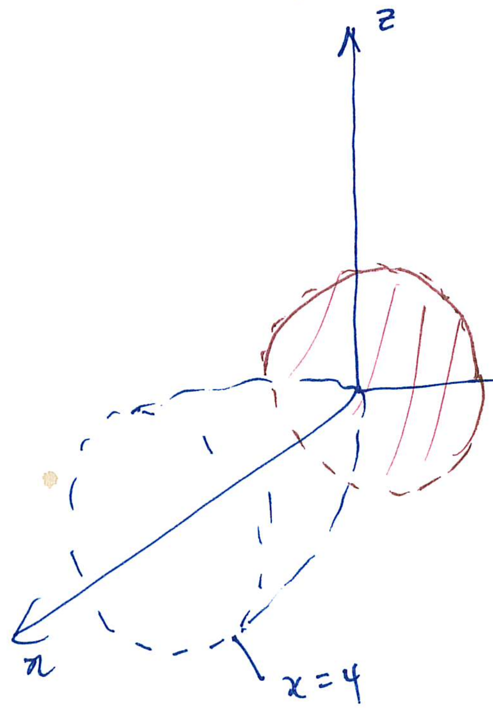
$$E = \{ (x,y,z) \mid (x,y) \in D,\ u_1(y,z) \le x \le u_2(y,z) \}.$$



$$\iiint_E f(x,y,z)\, dv = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\, dx \right] dA.$$

Example

Evaluate $\iiint_E x\, dV$, where E is bounded by $x = 4y^2 + 4z^2$ and the plane $x=4$



$$E = \{ (x,y,z) \mid 4y^2+4z^2 \le x \le 4 \}.$$

$$\frac{1}{2}\iint_D \left[\int_{4y^2+4z^2}^{4} x \, dx \right] dA$$

$$\frac{1}{2}\iint_D \left[4^2 - (4y^2+4z^2)^2 \right] dA$$

D is a circle of radius ≤ 1. $\cancel{4xy}$ $4y^2+4z^2 \le 4$

$y^2+z^2 \le 1.$

Write D in polar coordinates

$$\frac{1}{2}\int_0^{2\pi}\int_0^1 \left[4^2 - (4r^2)^2\right] r\,dr\,d\theta$$

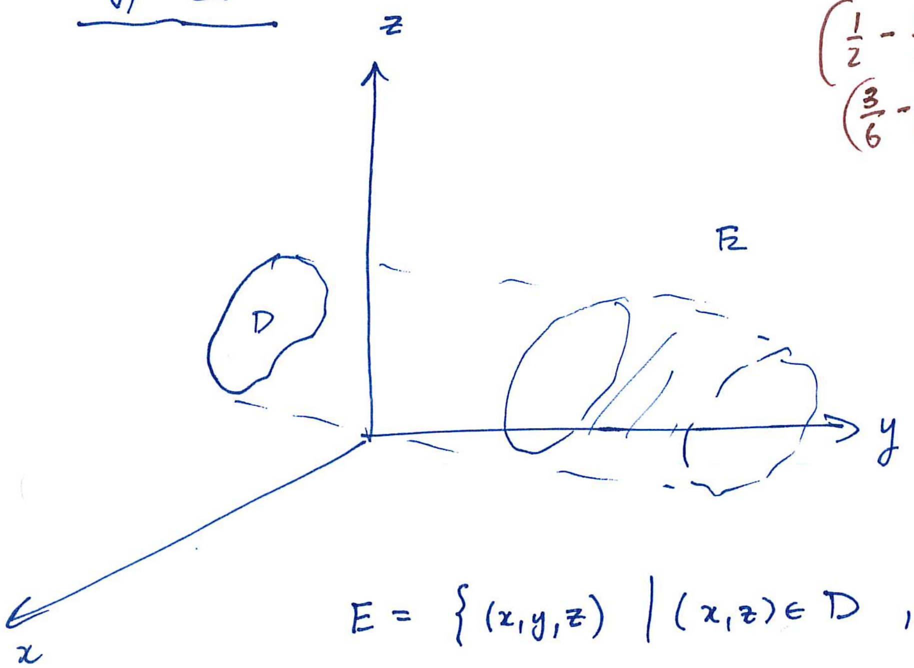
$$= 8\int_0^{2\pi}\int_0^1 (1-r^4)\, r\,dr\,d\theta \qquad = \frac{16\pi}{3}.$$

$$\int_0^1 r - r^5\,dr = \left.\frac{r^2}{2} - \frac{r^6}{6}\right|_0^1$$

$$\left(\frac{1}{2}-\frac{1}{6}\right)\cdot 8\cdot 2\pi$$

$$\left(\frac{3}{6}-\frac{1}{6}\right)\cdot 8\cdot 2\pi \quad \frac{2}{6}\cdot 16\pi = \frac{16\pi}{3}$$

Type III



$$E = \{(x,y,z) \mid (x,z)\in D,\ u_1(x,z) \le y \le u_2(x,z)\}$$

$$\iiint_E = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,dy\right] dA$$

Example ~~Find the volume of the~~

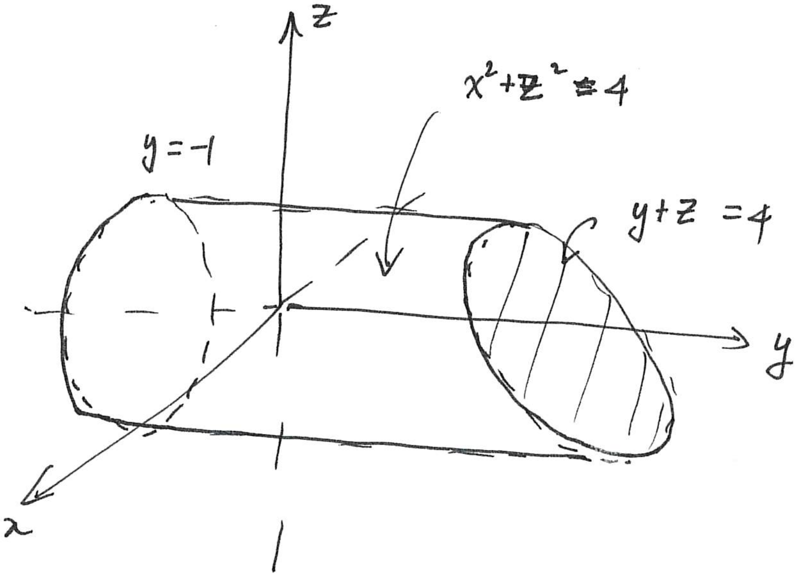
Set up $\iiint_E dV$ Find the volume enclosed by

E - solid enclosed by the cylinder $x^2+z^2=4$, and the planes $y=-1$ and $y+z=4$.

#7

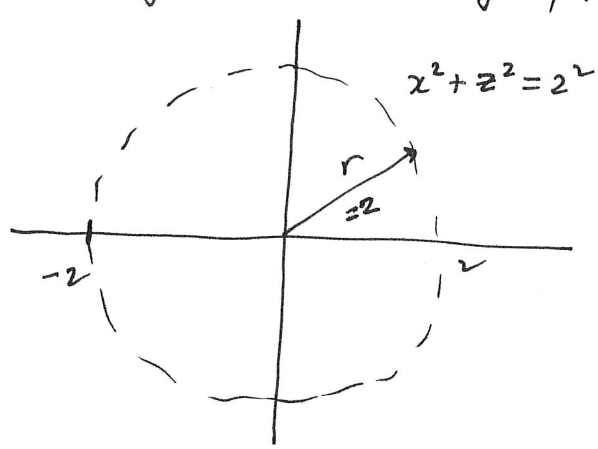
$\cancel{E = \{(x,y,z) \mid -1 \le y \le x^2+z^2 \le 4\}}$.

$E = \{(x,y,z) \mid -1 \le y \le 4-z \ , \ x^2+z^2 \le 4\}$



Projection into the yz plane

$\cancel{V = \int}$



$$V = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{-1}^{4-z} 1 \, dy \, dz \, dx$$

evaluate and then convert to polar

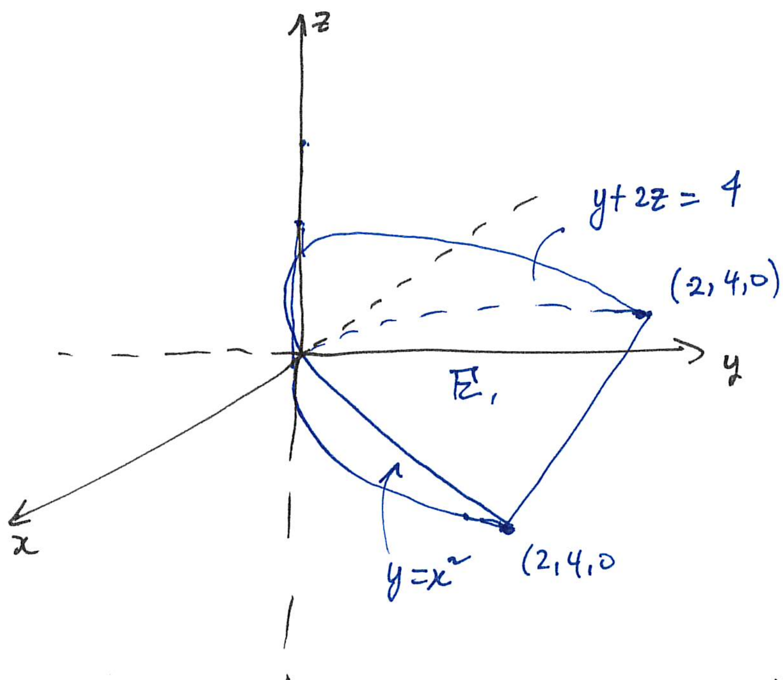
Set up

15.

$$\iiint_E f(x,y,z)\,dV$$

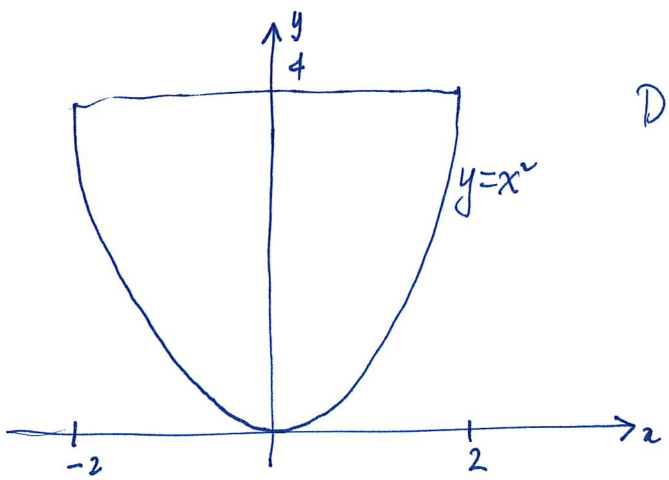
$y = x^2$
$y = 4$

$y = x^2, \quad z = 0, \quad y + 2z = 4$



D_1 - project onto the xy plane

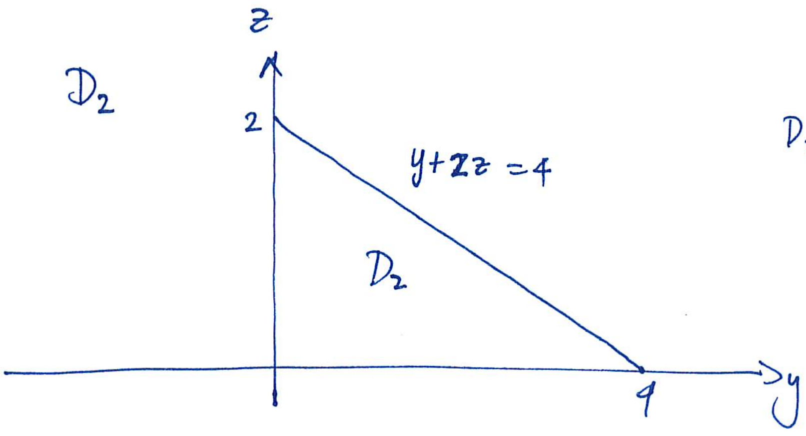
$\left[\; y + 2z = 4 \;\Rightarrow\; z = \frac{1}{2}(4 - y) \right]$



$D_1 = \{ (x,y) \mid -2 \le x \le 2,\ x^2 \le y \le 4 \}$

$$\int_{-2}^{2} \int_{x^2}^{4} \int_{0}^{\frac{1}{2}(4-y)} f(x,y,z)\,dz\,dy\,dx$$

D_2



$D_2 = \{ (y,z) \mid 0 \le y \le 4,\ 0 \le z \le 2 - \frac{1}{2}y \}$

$$\int_{0}^{4} \int_{0}^{2-\frac{y}{2}} \int_{-\sqrt{y}}^{\sqrt{y}} f(x,y,z)\,dx\,dz\,dy$$

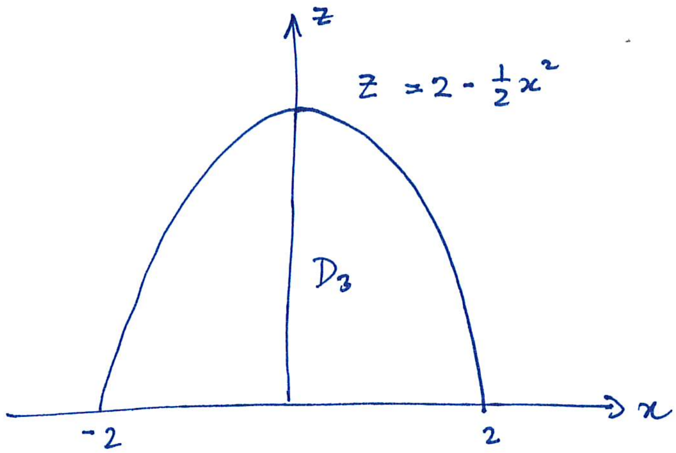
#8

$D_3 = \{(x,z) \mid -2 \le x \le 2,\quad 0 \le z \le 2 - \frac{x^2}{2}\}$

here $z = 2 - \frac{1}{2}y$, but $y = x^2$

so

$z = 2 - \frac{1}{2}x^2.$



D. $\iiint_E f(x,y,z)\,dV = \int_{-2}^{2}\int_{0}^{2-\frac{x^2}{2}}\int_{x^2}^{4-2z} f(x,y,z)\,dy\,dz\,dx$