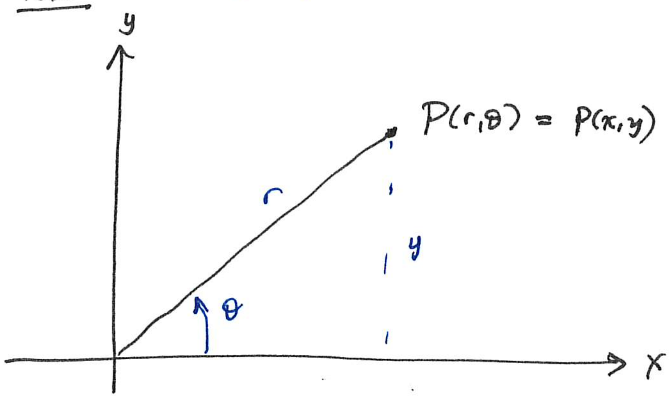
15.7 Triple Integrals in Cylindrical Coordinates

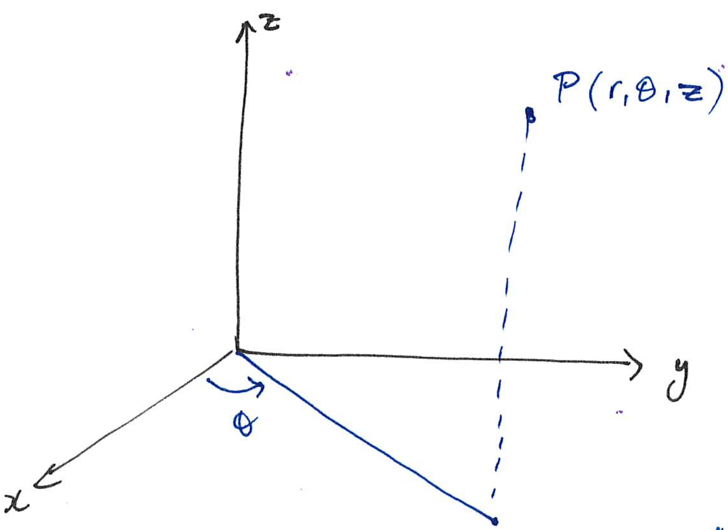
Polar (2D)



$x = r\cos\theta$ $\qquad$ $r^2 = x^2 + y^2$

$y = r\sin\theta$ $\qquad$ $\tan\theta = \dfrac{y}{x}$

Cylindrical Coordinates (3D)



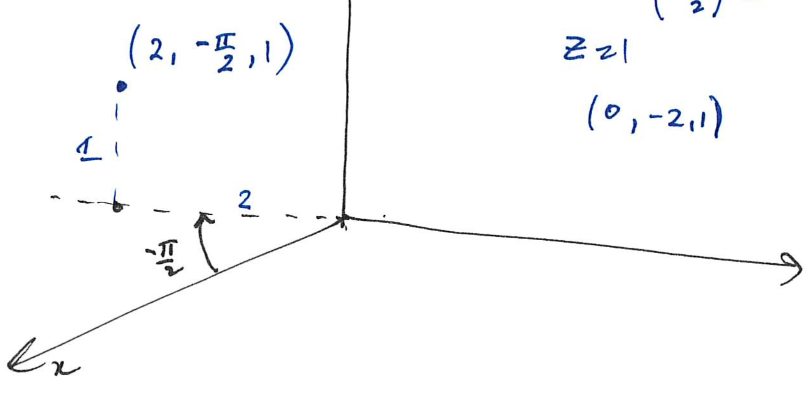
$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

Example #1

Plot $\left(2, -\frac{\pi}{2}, 1\right)$



In rectangular coordinates

$x = 2\cos\left(-\frac{\pi}{2}\right) = 0$ $\qquad$ $y = 2\sin\left(-\frac{\pi}{2}\right) = -2$

$z = 1$

$(0, -2, 1)$

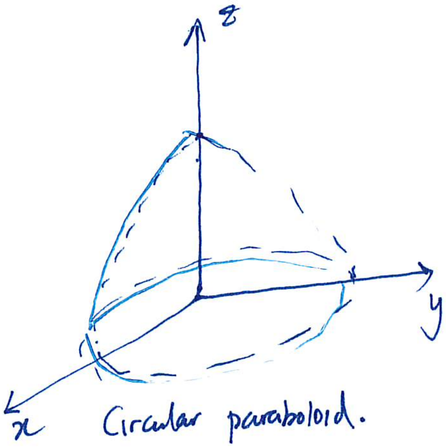
Example 15.8

Identify the surfaces

(a) $z = 4 - r^2$.

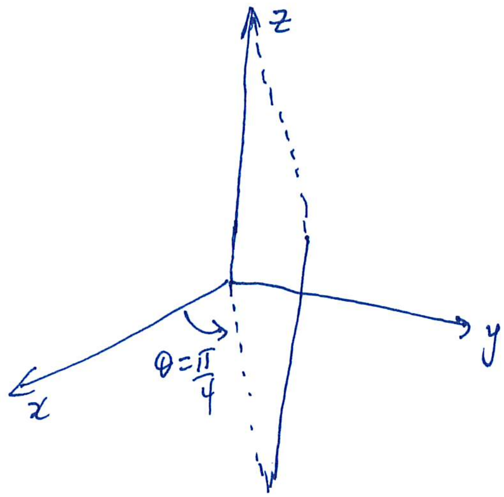
Recall that $r^2 = x^2 + y^2$

$z = 4 - (x^2 + y^2)$
$= 4 - x^2 - y^2$



Circular paraboloid.

(b) $\theta = \pi/4$

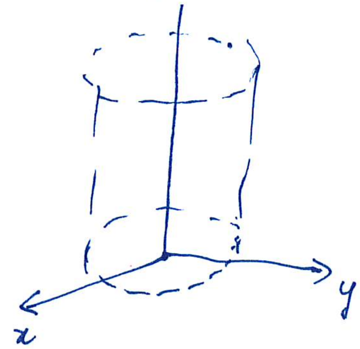


r and z may vary
vertical plane including
z-axis.

$r = 5$

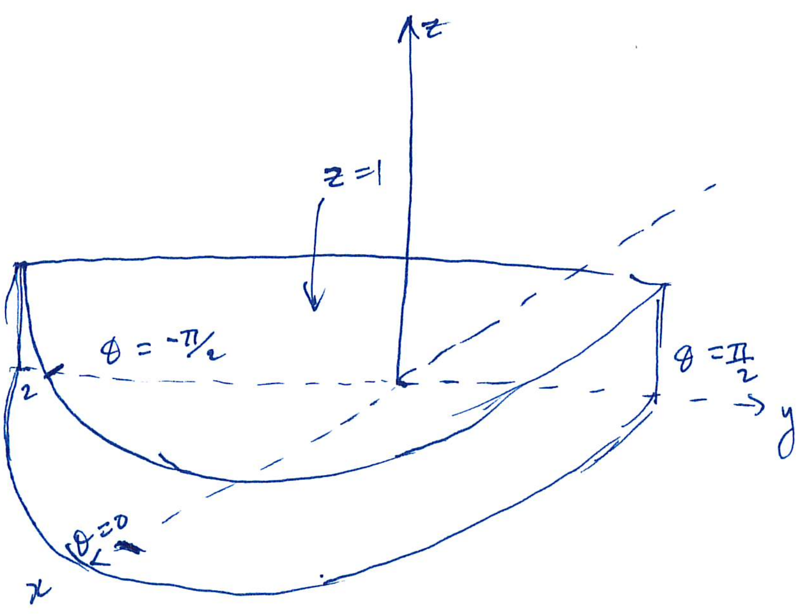
$r = \sqrt{x^2+y^2}$

$x^2 + y^2 = 5^2$



(b) Sketch the solid described by

$0 \le r \le 2$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, $0 \le z \le 1$



Solid circular cylinder with
radius of 2.

half cylinder.

Convert from rectangular to cylindrical $(-1,1,1)$

$r^2 = (-1)^2 + 1^2 = 2 \Rightarrow r = \sqrt{2}$; $\tan\theta = \frac{1}{-1} = -1$

$\theta = \frac{3\pi}{4} + 2n\pi$; $z = 1$.

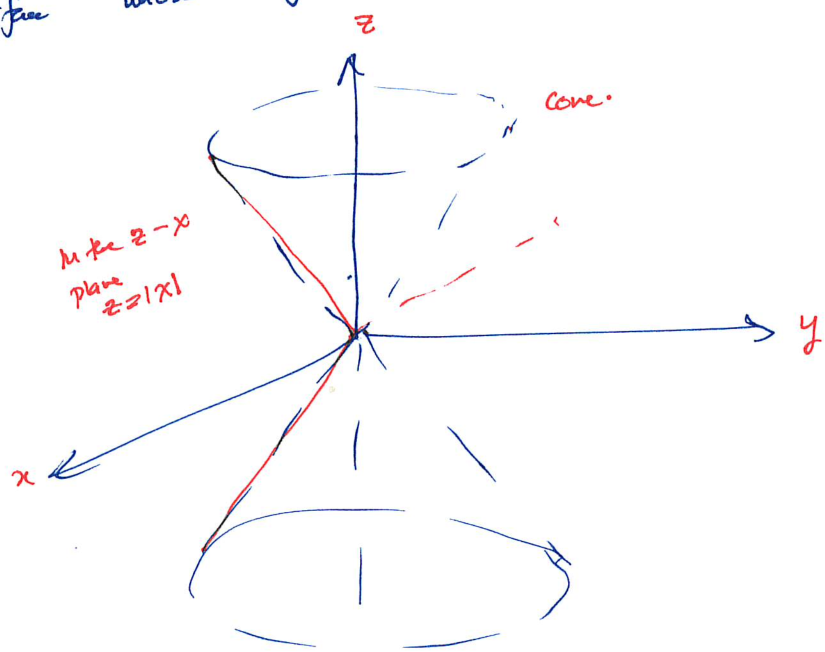
Why Cylindrical ?

- It is easy to describe regions that have symmetry about z axis

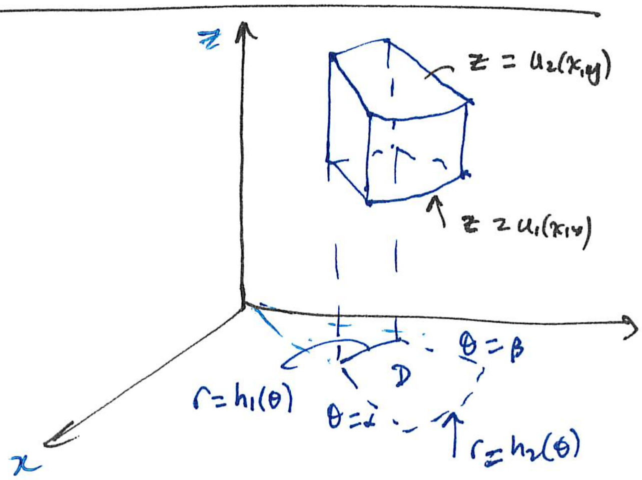
<u>example</u> ① $x^2+y^2 = c^2$, a cylinder becomes $r^2 = c^2 \Rightarrow r = c$.

② $\underline{z = r}$, $z = \sqrt{x^2+y^2}$ is the distance from the origin

Surface whose height



Evaluating Integrals in Cylindrical Coordinates



In polar!

$D = \{(r,\theta) \mid \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$

~~$u_1(x,y) \le z \le u_2(x,y)$~~

$\Uparrow$

$E = \{(x,y,z) \mid (x,y) \in D,\ u_1(x,y) \le z \le u_2(x,y)\}$

$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right] dA$$

Using the descriptn of D in polar coordinates.

$$\int = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta,\, r\sin\theta)}^{u_2(r\cos\theta,\, r\sin\theta)} f(r\cos\theta,\, r\sin\theta)\, r\,dz\,dr\,d\theta$$

Evaluate by switching to cylindrical

$$\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{\sqrt{x^2+y^2}}^{2} xz\,dz\,dx\,dy$$

z-coordinate

Above the cone $z=\sqrt{x^2+y^2}$ or $z=r$ below the plane $z=2$

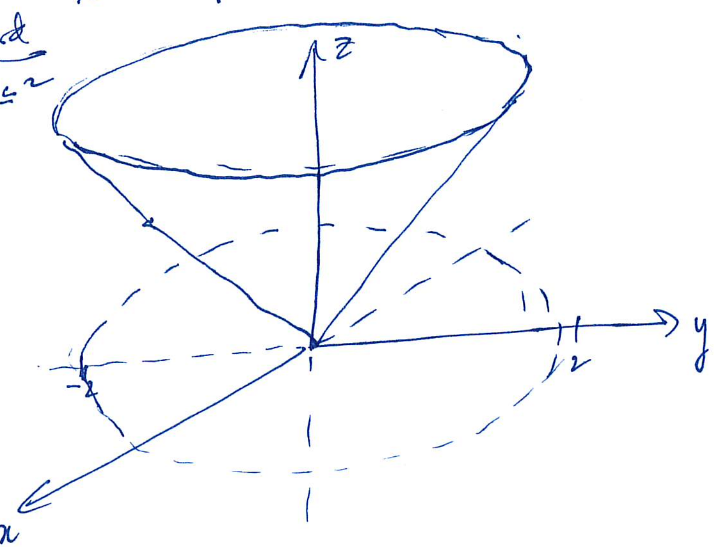
x-coordinates

$-\sqrt{4-y^2}\le x\le\sqrt{4-y^2}$

$x^2=4-y^2$ is a circle of radius 2 centered at $(0,0)$

y coord

$-2\le y\le 2$



$0\le\theta\le 2\pi$

$0\le r\le 2$

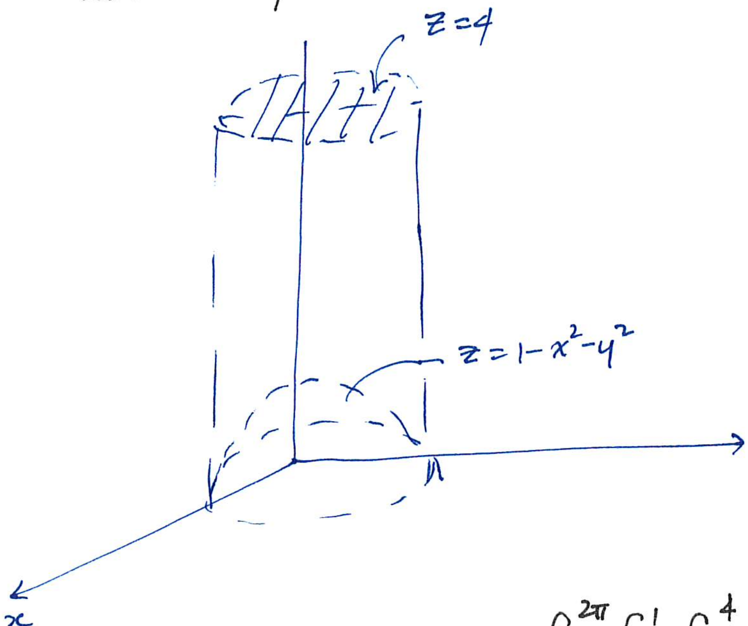
$r\le z\le 2$

$$\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{\sqrt{x^2+y^2}}^{2} xz\,dz\,dx\,dy = \int_{0}^{2\pi}\int_{0}^{2}\int_{r}^{2}(r\cos\theta)\,z\,r\,dz\,dr\,d\theta$$

$$= \int_{0}^{2\pi}\int_{0}^{2}\int_{r}^{2} r^2\cos\theta\, z\,dz\,dr\,d\theta$$

Example

Evaluate $\iiint_E \sqrt{x^2+y^2}\, dV$, where E lies within the cylinder $x^2+y^2=1$ below the plane $z=4$ above the paraboloid $z=1-x^2-y^2$



In cylindrical coordinates

$$E=\{(r,\theta,z)\mid 0\le\theta\le 2\pi,\ 0\le r\le 1,\ 1-r^2\le z\le 4.$$

Mention that $z=1-x^2-y^2$ is a parabola in $z-x$ and $z-y$ planes.

$$\iiint_E \sqrt{x^2+y^2}\,dV = \int_0^{2\pi}\int_0^1\int_{1-r^2}^4 r\cdot r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1\int_{1-r^2}^4 r^2\,dz\,dr\,d\theta$$

① $$r^2 z\Big|_{z=1-r^2}^{4} = 4r^2 - r^2(1-r^2). = 4r^2-r^2+r^4 = 3r^2+r^4$$

$$\int_0^{2\pi}\int_0^1 (3r^2+r^4)\,dr\,d\theta$$

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^1 (3r^2+r^4)\,dr\right)$$

$$= \left(\theta\Big|_0^{2\pi}\right)\left(\frac{3r^3}{3}+\frac{r^5}{5}\Big|_0^1\right)$$

$$=(2\pi)\left(1+\frac{1}{5}\right) = 2\pi\cdot\frac{6}{5} = \frac{12\pi}{5}$$