Initial Value Problems: Euler's method

$$y'(t) = 1 + \frac{y}{t}$$
$$y(1.0) = 1.0, \quad 1.0 \leq t \leq 6.0$$

Global error at each time step

t_i	y_ti	y(ti)	|y(ti) - y_ti|
1.0000	1.0000	1.0000	0
1.5000	2.0000	2.1082	0.1082
2.0000	3.1667	3.3863	0.2196
2.5000	4.4583	4.7907	0.3324
3.0000	5.8500	6.2958	0.4458
3.5000	7.3250	7.8847	0.5597
4.0000	8.8714	9.5452	0.6737
4.5000	10.4804	11.2683	0.7880
5.0000	12.1448	13.0472	0.9023
5.5000	13.8593	14.8761	1.0168
6.0000	15.6193	16.7506	1.1313

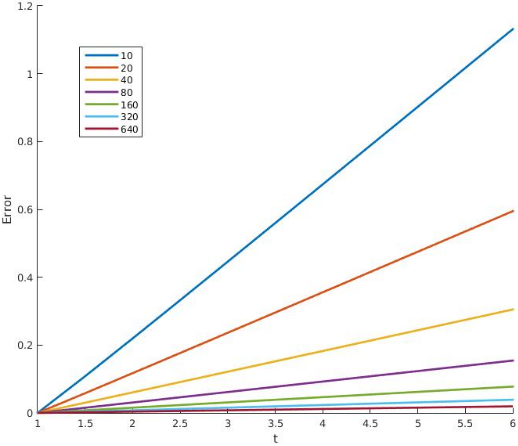
Euler's method: Convergence

$$y'(t) = 1 + \frac{y}{t}$$
$$y(1.0) = 1.0, \quad 1.0 \le t \le 6.0$$

n	approx sol	error @ t=6	Ratio
10	15.6193	1.1313	
20	16.1557	0.5948	1.9019
40	16.4456	0.3049	1.9507
80	16.5962	0.1544	1.9755
160	16.6729	0.0777	1.9878
320	16.7116	0.0389	1.9939
640	16.7311	0.0195	1.9970

Euler's method: Growth of error

$$y'(t) = 1 + \frac{y}{t}$$
$$y(1.0) = 1.0, \quad 1.0 \le t \le 6.0$$



Euler's method: Spreading of an epidemic

- *Variables*
  1. H - healthy individuals
  2. I - infected individuals
  3. D - Dead individuals
- *Assumptions*
  1. The disease is transmitted to healthy individuals at a rate proportional to HI

$$\frac{dH}{dt} = -cHI, \quad \text{where } c \text{ is the infection rate} \tag{1}$$

  2. The population changes due to death, birth or other causes is ignored because the population changes due to the epidemic are considered fast in comparison.

$$\frac{dI}{dt} = cHI - mI \tag{2}$$

$$\frac{dD}{dt} = mI \tag{3}$$

Euler's method: Spreading of an epidemic

$$\frac{dH}{dt} = -cHI, \quad (4)$$

$$\frac{dI}{dt} = cHI - mI \quad (5)$$

$$\frac{dD}{dt} = mI \quad (6)$$

Dividing (4) by (6) yields

$$\frac{dH}{dD} = -\frac{cHI}{mI} = -\frac{c}{m}H$$

with solution

$$H = H_0 e^{-\frac{c}{m}D}$$

where H_0 is the number of healthy individuals. The total population does not change therefore

$$\frac{d}{dt}(H + I + D) = 0 \rightarrow H + I + D = N \rightarrow I = N - H - D \quad (7)$$

Plugging (7) into (6) and using (5) yields

$$\frac{dD}{dt} = m[N - D - H_0 e^{-\frac{c}{m}D}]$$

Euler's method: Spreading of an epidemic

We can solve the IVP

$$\frac{dD}{dt} = m[N - D - H_0 e^{-\frac{c}{m}D}] \quad (8)$$

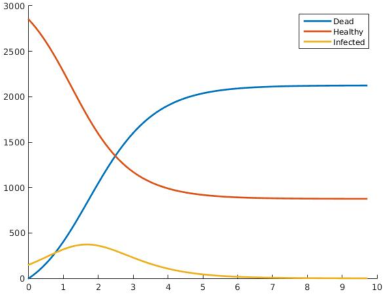
$$D(0) = 0 \quad (9)$$

using Euler's method to determine how long it takes for the epidemic to run its course.

```
N= 3000; %initial population
H_0 = 3000-300; % (initial healthy)
D_0 = 0; %initially nobody is dead.
m=1.8; % mortality rate
c=0.001;% transmission rate
h =0.1; %time step in weeks
%Using Euler's start @ t=0;
approx_sol_D =[];
approx_sol_H =[];
approx_sol_I =[];
. . .
```

Euler's method: Spreading of an epidemic

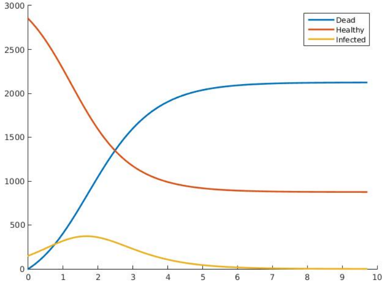
```
N= 3000; %initial population
H_0 = 3000-150; % (initial healthy)
D_0 = 0; %initially nobody is dead.
m=1.8; % mortality rate
c=0.001;% transmission rate
h =0.1; %time step in weeks
```




```
Number of people dead at 10 weeks is 2124
```

Euler's method: Spreading of an epidemic

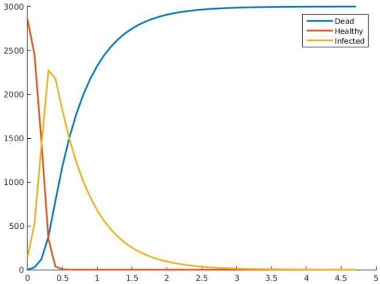
```
N= 3000; %initial population
H_0 = 3000-300; % (initial healthy)
D_0 = 0; %initially nobody is dead.
m=1.8; % mortality rate
c=0.001;% transmission rate
h =0.1; %time step in weeks
```



Number of people dead at 8 weeks is 2207

Euler's method: Spreading of an epidemic

```
N= 3000; %initial population
H_0 = 3000-150; % (initial healthy)
D_0 = 0; %initially nobody is dead.
m=1.8; % mortality rate
c=0.01;% transmission rate
h =0.1; %time step in weeks
```



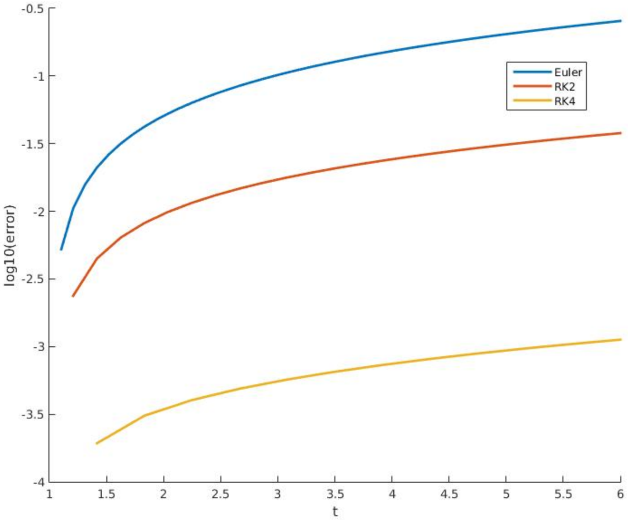
Number of people dead at 5 weeks is 3000

Euler's method: Computing orbit trajectories

- The story of 3 African American mathematicians working as "computers" for the NSA during the space program.
- They used Eulers method to solve ODEs modelling the flight of a capsule through space!

Numerical efficiency

`h_euler = 0.125, h_rk2 = 0.25, h_rk4 = 0.5`



Consistency of multistep methods

Theorem

The local truncation error of the multistep method

$$\sum_{l=0}^{m} a_l y_{k+l} = h \sum_{l=0}^{m} b_l f(t_{k+l}, y_{k+l}) \tag{10}$$

is of order $p \geq 1$ if and only if

$$\sum_{l=0}^{m} a_l = 0 \ \mathit{and} \ \sum_{l=0}^{m} l^j a_l = j \sum_{l=0}^{m} l^{j-1} b_l, \quad j = 1, \ldots, p.$$

Consistency of multistep methods

Proof.

$$
\begin{aligned}
\tau(t,y,h) &= \frac{1}{h}\sum_{l=0}^{m} a_l y(t+lh) - \sum_{l=0} b_l y'(t+lh) \\
&= \frac{1}{h}\sum_{l=0}^{m} a_l \sum_{j=0}^{p} \frac{(lh)^j}{j!} y^{(j)}(t) - \sum_{l=0}^{m} b_l \sum_{j=0}^{p-1} \frac{(lh)^j}{j!} y^{(j+1)}(t) + \mathcal{O}(h^p) \\
&= \frac{1}{h}\left(\sum_{l=0}^{m} a_l\right) y(t) + \sum_{j=1}^{p} \frac{h^{j-1}}{j!}\left(\sum_{l=0}^{m} l^j a_l\right) y^{(j)}(t) \\
&- \sum_{j=0}^{p-1} \frac{h^j}{j!}\left(\sum_{l=0}^{m} l^j b_l\right) y^{(j+1)}(t) + \mathcal{O}(h^p) \\
&= \frac{1}{h}\left(\sum_{l=0}^{m} a_l\right) y(t) + \sum_{j=1}^{p} \frac{h^{j-1}}{j!}\left(\sum_{l=0}^{m} l^j a_l - j\sum_{l=0}^{m} l^{j-1} b_l\right) y^j(t) \\
&+ \mathcal{O}(h^p).
\end{aligned}
$$