Math 251 **Name (Print):** ____________________
Fall 2018
Exam 2
11/08/2018
Instructor: Dr. Prince Chidyagwai
Time Limit: 1 hour 15min

This exam contains 9 pages (including this cover page) and 9 problems including a BONUS problem worth 5 points. Check to see if any pages are missing. Enter all requested information on the top of this page.
You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	6	
2	6	
3	8	
4	30	
5	10	
6	16	
7	14	
8	10	
9	5	
Total:	105	

1. Let

$$f(x) = \begin{cases} x^2+1, & \text{if } x<1 \\ x+1 & \text{if } x \geq 1 \end{cases}$$

(a) (2 points) Find $f'(x)$

$$f'(x) = \begin{cases} 2x, & x<1 \\ 1, & x \geqslant 1 \end{cases}$$

(b) (4 points) Is the function f differentiable at $x=1$? Explain.

No, $f'_+(1) = 1$ and $f'_-(1) = 2$

So $f'(1)$ is not defined.

2. (6 points) Suppose $f(x) = 2e^x + 3x + 5x^3$. Show that f has no tangent line with a slope of 2.

$f'(x) = 2e^x + 3 + 15x^2$

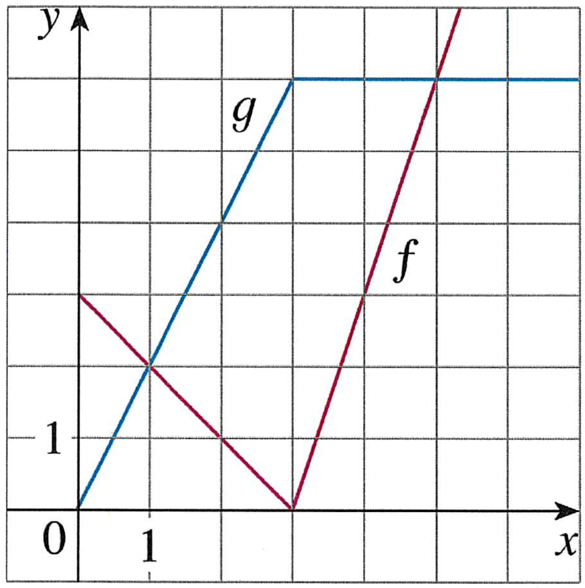
$f'(x) > 3$ because $e^x > 0$ and $15x^2 \geqslant 0$

so f cannot have a tangent line of slope 2.

3. If f and g are functions whose graphs are shown below, let

$$P(x) = f(x)g(x), Q(x) = \frac{f(x)}{g(x)} \text{ and } C(x) = f(g(x))$$

find



(a) (2 points) $P'(2)$

$P'(x) = f'(x)g(x) + g'(x)f(x)$ so

$P'(2) = f'(2)g(2) + g'(2)f(2)$ ← [use the lines to find the values of each and plug in!]

↓

(b) (3 points) $Q'(2)$

$$Q'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2} \Rightarrow Q'(2) = \frac{g(2)f'(2) - f(2)g'(2)}{(g(2))^2}$$

(c) (3 points) $C'(2)$

$C'(x) = f'(g(x)) \cdot g'(x)$

$C'(2) = f'(g(2)) \cdot g'(2)$

$= f'(4) \cdot g'(2)$

4. Find the derivative for each of the following functions. You do NOT need to simplify your answers.

(a) (6 points) $y = (x+1)^{\frac{2}{3}}(2x^2-1)^3$

$$y' = \frac{2}{3}(x+1)^{-\frac{1}{3}} \cdot (2x^2-1)^3 + (x+1)^{\frac{2}{3}} \cdot 3(2x^2-1)^2 \cdot 4x$$

(b) (6 points) $y = \dfrac{e^5 + \sin(5x)}{e^{-x}}$

$$y' = \frac{e^{-x}\left(5\cos(5x)\right) - \left(e^5 + \sin(5x)\right)\cdot\left(-e^{-x}\right)}{(e^{-x})^2}$$

(c) (6 points) $y = 2^{\sin^{-1}(x)}$

$$y' = 2^{\sin^{-1}(x)} \cdot \ln(2) \cdot \frac{d}{dx}\left(\sin^{-1}(x)\right)$$

$$= 2^{\sin^{-1}(x)} \cdot \ln(2) \cdot \frac{1}{\sqrt{1-x^2}}.$$

(d) (6 points) $y = 2x^{\tan x}$

$\ln(AB) = \ln(A) + \ln(B)$

$$\ln(y) = \ln\left(2x^{\tan x}\right)$$
$$= \ln(2) + \ln\left(x^{\tan x}\right)$$
$$= \ln(2) + \tan(x)\cdot\ln(x)$$

Taking the derivative implicitly

$$\frac{1}{y}\cdot y' = \tan(x)\cdot\frac{1}{x} + \ln(x)\cdot\sec^2(x)$$
$$y' = y\left[\frac{\tan(x)}{x} + \ln(x)\cdot\sec^2(x)\right]$$

(e) (6 points) $y = \ln(\cos^2(x))$

$$y' = \frac{1}{\cos^2(x)}\cdot\frac{d}{dx}\left(\cos^2(x)\right)$$
$$= \frac{1}{\cos^2(x)}\cdot 2\cos(x)\cdot(-\sin(x))$$

5. (10 points) Find the equation of the tangent line to the curve $\sin(x+y) = 2x - 2y$ at the point (π, π)

$$\sin(x+y) = 2x - 2y$$
$$\cos(x+y)\left(1 + \frac{dy}{dx}\right) = 2 - 2\frac{dy}{dx}$$
$$\cos(x+y) + \cos(x+y)\frac{dy}{dx} = 2 - 2\frac{dy}{dx}$$
$$\left(\cos(x+y) + 2\right)\frac{dy}{dx} = 2 - \cos(x+y)$$
$$\frac{dy}{dx} = \frac{2 - \cos(x+y)}{\cos(x+y)+2}, \quad \frac{dy}{dx} @ (\pi,\pi) = \frac{2 - \cos(2\pi)}{\cos(2\pi)+2} = \frac{1}{3}$$

Equation of tangent line

$$y - \pi = \frac{1}{3}(x - \pi)$$

6. (a) (4 points) The cost of production of a x cars in dollars is given by a function $y = C(x)$. Give the interpretation for $C'(100)$ including the units.

$C'(100)$ is the additional cost of producing one more car i.e the cost of the 101^{st} car.

(b) (4 points) Let $n = f(t)$ be the number of bacteria in a petri dish at time t. What is the signicance of $f'(t)$?

$f'(t)$ - Rate of growth of bacteria population.

(c) (8 points) If \$1000 is invested at in interest rate of $r\%$ compounded monthly, the account balance after t years is given by

$$B = 1000\left(1 + \frac{r}{12}\right)^{12t}$$

1. Find the instantaneous rate of change of the balance B with respect to time t assuming the interest rate r is a constant. Give the appropriate units.

Notice that in this case we have an exponential function

Recall that

$\frac{d}{dt}(a^t) = a^t \ln(a)$

$$\frac{dB}{dt} = 1000\left(1+\frac{r}{12}\right)^{12t} \ln\left(1+\frac{r}{12}\right) \cdot 12$$

Chain Rule!

2. Find the instantaneous rate of change of the balance B with respect to the interest rate r assuming t is a constant.

Now we have an exponential function.

$$\frac{dB}{dr} = 12t \cdot 1000\left(1+\frac{r}{12}\right)^{12t-1}$$

7. The position of a particle moving in a straight line is given by $s(t) = t^3 - 6t^2 + 9t$.

(a) (3 points) Find the velocity at time t.

$$v(t) = s'(t) = 3t^2 - 12t + 9$$

(b) (5 points) When is the particle at rest?

When $v(t) = 0 \iff 3t^2 - 12t + 9 = 0$

$$3(t^2 - 4t + 3) = 0$$

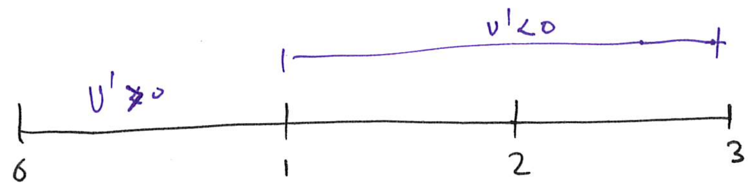
$$3(t-1)(t-3) = 0$$

$$t = 1,\ t = 3$$

(c) (2 points) Find the acceleration of the particle at time t.

$$a(t) = s''(t) = 6t - 12.$$

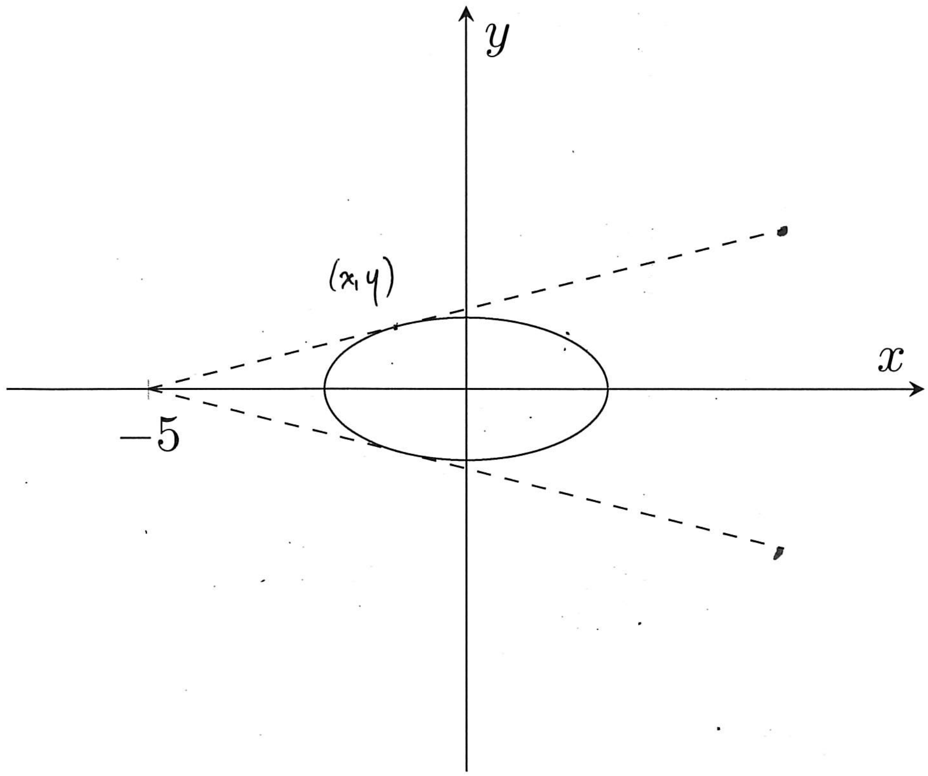
(d) (4 points) Find the distance travelled by the particle in the time interval $[0, 3]$.



$$s(1) - s(0) + |s(3) - s(1)|$$

$$= \underline{8m}$$

8. (10 points) The ellipse with equation $x^2 + 4y^2 = 5$ has two tangent lines that pass through the point $(-5, 0)$ as shown in the plot below.



Find the coordinates of the points of intersection of the tangent lines and the ellipse.

Let (x,y) be the point of intersection.

The slope of the tangent line can be computed in 2 ways

① $x^2 + 4y^2 = 5$ so $2x + 8y \frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-2x}{8y} = -\frac{x}{4y} = m$

② We also know that the tangent line passes through (x,y) & $(-5,0)$ $\frac{y-0}{x+5} = m$

So $\frac{y}{x+5} = -\frac{x}{4y}$ ... (i) $\Rightarrow 4y^2 = -x^2 - 5x$

the point (x,y) is on the ellipse so $x^2 + 4y^2 = 5$ ... (ii)

from (ii) $4y^2 = 5 - x^2$ so $-x^2 - 5x = 5 - x^2 \Rightarrow -5x = 5$ so $x = -1$

plug into $x^2 + 4y^2 = 5 \Rightarrow 4y^2 = 4 \Rightarrow y = \pm 1$ so $(x,y) = (1, \pm 1)$

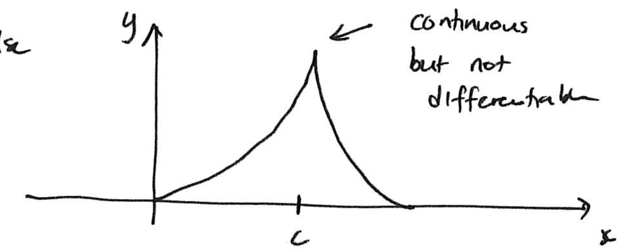
9. (5 points) (BONUS) **Decide whether each of the following statements are true/false. Explain your reasoning**

(a) If $y = e^2$ then $y' = 2e$.

False e^2 is a constant

(b) If f is continuous at $x = c$ then f is differentiable at $x = c$

False



(c) If f is differentiable then $\frac{d}{dx}(f(\sqrt{x})) = \frac{f'(x)}{2\sqrt{x}}$

False Proper chain Rule should be

$$\frac{d}{dx} f(\sqrt{x}) = f'(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}}.$$