1. Let \( f(x) = x - p - q\cos(x) \), we want to find an interval on which \( f \) changes sign and use the intermediate value theorem to show that there is a root. Since \( \cos(\frac{\pi}{2} + n\pi) = 0 \), \( f(\frac{\pi}{2} + n\pi) = \frac{\pi}{2} + n\pi - p \) for any integer \( n \). Since \( p \) is fixed there exists an integer \( n^* \) such that

\[
\frac{\pi}{2} + n^*\pi < p < \frac{\pi}{2} + (n^* + 2)\pi
\]

Our function is the sum of linear and trig functions therefore \( f \) is continuous and

\[
f\left(\frac{\pi}{2} + n^*\pi\right) = \frac{\pi}{2} + n^*\pi - p < 0 \quad (1)
\]

\[
f\left(\frac{\pi}{2} + (n^* + 2)\right) = \frac{\pi}{2} + (n^* + 2) - p > 0 \quad (2)
\]

therefore by IVT \( f \) has a root on \( \left(\frac{\pi}{2} + n^*\pi, \frac{\pi}{2} + (n^* + 2)\right) \)

2. The size of each interval is reduced by a factor of 2 at each step so after \( k \) steps we have an interval of size \( \frac{|b-a|}{2^k} \). To ensure that the approximate root is within \( \epsilon \) of the true solution we need

\[
\frac{|b-a|}{2^k} \leq 2\epsilon \iff k \geq \log_2\left(\frac{b-a}{2\epsilon}\right) - 1
\]

so in our case \( b-a = 3 \) and \( \epsilon = 10^{-9} \) so \( k \geq \log_2\left(\frac{3}{2 \cdot 10^{-9}}\right) - 1 \)

3. The point of this exercise is numerically confirm that

\[
\lim_{n \to \infty} \frac{|x_{n+1} - x_*|}{|x_n - x_*|^2} = \left| \frac{f''(x_*)}{2f'(x_*)} \right|
\]

4. From the definition of Newton’s method plug in the appropriate values \( f'(x_0) = 1 \).

5. From the definition of Secant method plug in the appropriate values \( f(x_0) = 16 \).

6. The Bisection method cannot be used to find the root of \( f(x) = \sin(x) + 1 \) because \( f(x) \leq 0 \) for all \( x \). Newton’s method can be used for this problem but the convergence degrades to linear because the derivative at the roots is zero.

7. (a). In this case \( f(x) = x^2 - a \) and \( f'(x) = 2x \) so we can implement Newton’s method as

\[
x_{n+1} = x_n - \frac{(x_n^2 - a)}{2x_n}
\]

\[
= x_n - \frac{1}{2} \left( x_n - \frac{a}{x_n} \right)
\]

\[
= \frac{1}{2} \left( x_n + \frac{a}{x_n} \right)
\]
(b). In class we showed that

\[
\frac{x_n - x_{n+1}}{(x_n - x_{n})^2} = -\frac{f''(\xi_n)}{2f'(x_n)} \quad \text{for } \xi_n \in (x, x_n)
\]

in our case \(x_n = \sqrt{a}\) and \(f'(x) = 2x, f''(x) = 2\) therefore

\[
\frac{\sqrt{a} - x_{n+1}}{(\sqrt{a} - x_n)^2} = -\frac{2}{2 \cdot 2x_n} = \frac{1}{2x_n}
\]