Homework 2 - Solutions

October 14, 2018

1. (a)(i) If \( x^* \) is a fixed point of \( x_{n+1} = \varphi(x_n) \) then \( x^* = \varphi(x^*) = x^* - \frac{f(x^*)}{f'(x^*)} \) which yeilds

\[
x^* = x^* - \frac{f(x^*)}{f'(x^*)} \implies \frac{f(x^*)}{f'(x^*)} = 0
\]

so \( f(x^*) = 0 \) provided \( f'(x^*) \neq 0 \). Therefore \( x^* \) is a root of \( f(x^*) = 0 \).

(a)(ii). If \( f'(x^*) \neq 0 \), then \( \varphi'(x^*) = 0 \). We compute

\[
\varphi'(x) = 1 - \left\{ \frac{f'(x)f'(x) - f(x)f''(x)}{[f'(x)]^2} \right\}
\]

(1)

\[
\frac{f(x)f''(x)}{[f'(x)]^2}
\]

(2)

Therefore \( \varphi'(x^*) = \frac{f(x^*)f''(x)}{[f'(x^*)]^2} = 0 \) because \( f(x^*) = 0 \).

(a)(iii). If \( f'(x^*) \neq 0 \) then \( \varphi''(x^*) \neq 0 \). Compute

\[
\varphi''(x) = \frac{[f'(x)]^2 \left\{ f(x)f'''(x) + f''(x)f'(x) \right\} - [f(x)f''(x)]2f'(x)f''(x)}{[f'(x)]^4}
\]

(3)

Evaluating \( \varphi''(x) \) at \( x^* \) and using the fact that \( f(x^*) = 0 \) yeilds

\[
\varphi''(x^*) = \frac{f''(x^*)f'(x^*)}{f'(x^*)^2} \neq 0
\]

since \( f''(x^*) \neq 0 \) and \( f'(x^*) \neq 0 \).

(b) A Taylor expansion of \( \varphi(x) \) about \( x^* \) yeilds

\[
\varphi(x) = \varphi(x^*) + (x - x^*)\varphi'(x^*) + \frac{1}{2}(x - x^*)^2\varphi''(\xi_k), \text{ for } \xi_k \in (x, x^*)
\]

Then evaluating at \( x = x_k \) and using the fact that \( \varphi'(x^*) = 0 \) (from (a)) yeilds

\[
\varphi(x_k) = \varphi(x^*) + \frac{1}{2}(x_k - x^*)^2\varphi''(\xi_k), \text{ for } \xi_k \in (x_k, x^*).
\]

Recall that \( x_{k+1} = \varphi(x_k) \) and \( x^* = \varphi(x^*) \) above, we obtain

\[
x_{k+1} = x^* + \frac{1}{2}(x_k - x^*)^2\varphi''(\xi_k)
\]

so

\[
x^* - x_{k+1} = -\frac{1}{2}(x_k - x^*)^2\varphi''(\xi_k)
\]
Then taking the limit as \( k \to \infty \) and noting that \( \xi_k \) converges to \( x_* \) we get
\[
\lim_{n \to \infty} \frac{x_* - x_{n+1}}{(x_* - x_k)^2} = -\frac{1}{2} \varphi''(x_*)
\]

This proves that Newton’s method converges with a quadratic rate because \( \varphi''(x_*) \neq 0 \).

2. We have \( x = c + d \cos(x) \equiv \varphi(x) \). Here we can use the contraction mapping theorem

**Theorem 1** Assume \( \varphi(x) \) and \( \varphi'(x) \) are continuous on \([a, b]\) and \( a \leq x \leq b \implies a \leq \varphi(x) \leq b \). Further assume that \( \max_{a \leq x \leq b} |\varphi'(x)| < 1 \)

(a) There exists a unique solution, \( x_* \) of \( x = \varphi(x) \) in \([a, b]\).

(b) For any \( x_0 \in [a, b] \) the sequence \( \{x_n\} \) converges to \( x_* \).

(c) The convergence is linear, i.e \( \lim_{n \to \infty} \frac{x_* - x_{n+1}}{x_* - x_n} = \varphi'(x_*) \).

In our case \( \varphi(x) = c + d \cos(x) \) and \( \varphi'(x) = -d \sin(x) \) are both continuous for all \( x \). If we choose \( a = c - |d| \) and \( b = c + |d| \), then if \( x \in [a, b], \varphi(x) = c + d \cos(x) \in [a, b] \) because \(-1 \leq \cos(x) \leq 1\). This gives us existence of the root. For uniqueness and convergence we note that \( |\varphi'(x)| \leq d < 1 \) therefore by the contraction mapping theorem the convergence is at least linear.

3. For \( x_{n+1} = 2 - (1 + c)x_n + cx_n^3 \), to obtain quadratic convergence we need \( \varphi'(x_*) = 0 \) but \( \varphi''(x_*) \neq 0 \). So we compute \( \varphi'(1) = -(1 + c) + 3c = 0 \implies c = \frac{1}{2} \). We can check \( \varphi''(1) = 6c \neq 0 \) for our choice of \( c \) therefore the convergence is exactly quadratic and not higher.

4. To determine the machine epsilon, \( \epsilon \) take the difference between 1 and the floating point number after 1, \( 1.001 \times 2^0 \) to obtain \( \epsilon = \frac{1}{8} \). The smallest positive number is \( 1.000 \times 2^{-1} = \frac{1}{2} \) and the largest number is \( 1.111 \times 2^1 = \frac{15}{4} \).