5.7 Hermite and Hermite Cubic Interpolation

1. Show that the polynomials \( H_i \) and \( \hat{H}_i \) defined by

\[
H_i(x) = [1 - 2L'_{n,i}(x_i)(x - x_i)]L^2_{n,i}(x)
\]
\[
\hat{H}_i(x) = (x - x_i)L^2_{n,i}(x),
\]

where \( L_{n,i} \) is the Lagrange polynomial associated with the point \( x = x_i \) satisfy the relations

\[
H_i(x_j) = \begin{cases} 1, & i = j \\ 0, & \text{otherwise} \end{cases} \quad \hat{H}_i(x_j) = 0
\]
\[
H'_i(x_j) = 0 \quad \hat{H}'_i(x_j) = \begin{cases} 1, & i = j \\ 0, & \text{otherwise} \end{cases}
\]

For \( i \neq j \),

\[
H_i(x_j) = [1 - 2L'_{n,i}(x_i)(x_j - x_i)]L^2_{n,i}(x_j)
\]
\[
= [1 - 2L'_{n,i}(x_i)(x_j - x_i)](0)^2 = 0
\]

and

\[
\hat{H}_i(x_j) = (x_j - x_i)L^2_{n,i}(x_j)
\]
\[
= (x_j - x_i)(0)^2 = 0.
\]

Moreover,

\[
H_i(x_i) = [1 - 2L'_{n,i}(x_i)(x_i - x_i)]L^2_{n,i}(x_i)
\]
\[
= [1 - 2L'_{n,i}(x_i)(0)](1)^2 = 1
\]

and

\[
\hat{H}_i(x_i) = (x_i - x_i)L^2_{n,i}(x_i)
\]
\[
= (0)(1)^2 = 0.
\]

For the derivatives, note that

\[
H'_i(x) = 2[1 - 2L'_{n,i}(x_i)(x - x_i)]L_{n,i}(x)L'_n,x_i(x) - 2L^2_{n,i}(x)L'_{n,i}(x_i)
\]
\[
\hat{H}'_i(x) = L^2_{n,i}(x) + 2(x - x_i)L_{n,i}(x)L'_{n,i}(x).
\]
Thus, for \( i \neq j \),
\[
H'_i(x_j) = 2[1 - 2L'_{n,i}(x_j)(x_j - x_i)]L_{n,i}(x_j)L'_{n,i}(x_j) - 2L^2_{n,i}(x_j)L''_{n,i}(x_i)
\]
\[
= 2[1 - 2L'_{n,i}(x_j)(x_j - x_i)](0)L'_{n,i}(x_j) - 2(0)^2L''_{n,i}(x_i) = 0
\]
and
\[
\dot{H}'_i(x_j) = L^2_{n,i}(x_j) + 2(x_j - x_i)L_{n,i}(x_j)L'_{n,i}(x_j)
\]
\[
= (0)^2 + 2(x_j - x_i)(0)L'_{n,i}(x_j) = 0.
\]
Moreover,
\[
H'_i(x_i) = 2[1 - 2L'_{n,i}(x_i)(x_i - x_i)]L_{n,i}(x_i)L'_{n,i}(x_i) - 2L^2_{n,i}(x_i)L''_{n,i}(x_i)
\]
\[
= 2[1 - 2L'_{n,i}(x_i)(0)](1)L'_{n,i}(x_i) - 2(1)^2L''_{n,i}(x_i)
\]
\[
= 2L'_{n,i}(x_i) - 2L''_{n,i}(x_i) = 0
\]
and
\[
\dot{H}'_i(x_i) = L^2_{n,i}(x_i) + 2(x_i - x_i)L_{n,i}(x_i)L'_{n,i}(x_i)
\]
\[
= (1)^2 + 2(0)(1)L'_{n,i}(x_i) = 1.
\]

2. Let \( f \) be continuously differentiable \( 2n + 2 \) times on \([a, b]\), and let \( x_0, x_1, x_2, \ldots, x_n \) be \( n + 1 \) distinct points from \([a, b]\). Provide the details of the proof that for each \( x \in [a, b] \), there exists a \( \xi \in [a, b] \) such that
\[
f(x) = P(x) + \frac{f^{(2n+2)}(\xi)}{(2n+2)!} \prod_{i=0}^{n} (x - x_i)^2,
\]
where \( P \) is the Hermite interpolating polynomial.

First note that since \( P(x_i) = f(x_i) \) by the interpolation conditions and since the term involving \( f^{(2n+2)} \) contains the factor \( (x - x_i) \), the error formula holds for each abscissa, \( x = x_i \). For all other \( x \in [a, b] \), consider the auxiliary function
\[
g(t) = f(t) - P(t) - [f(x) - P(x)] \prod_{i=0}^{n} \frac{(t - x_i)^2}{(x - x_i)^2}.
\]

By hypothesis, \( f \) has \( 2n+2 \) continuous derivatives on \((a, b)\). Since \( P \) and \( \prod_{i=0}^{n} \frac{(t-x_i)^2}{(x-x_i)^2} \) are polynomials in \( t \), they possess infinitely many continuous derivatives on \((a, b)\). By construction, then, \( g \) has \( 2n + 2 \) continuous derivatives on \((a, b)\). Furthermore,
\[
g(x_j) = f(x_j) - P(x_j) - [f(x) - P(x)] \prod_{i=0}^{n} \frac{(x_j - x_i)^2}{(x - x_i)^2}
\]
\[
= f(x_j) - P(x_j) - 0 = 0
\]