3.7 Rates of change (Applications)

\[ \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \quad \text{(Instantaneous rate of change of y w.r.t. x)} \]

**Physics**

1. \( S(t) \rightarrow V(t) \rightarrow A(t)\)

2. **Non-homogeneous rod**

   - If a rod is homogeneous, \( p = \frac{m}{l} \text{ kg/metre} \)
   - If the rod is not homogeneous but \( m = f(x) \)

   \[ \text{Average density} = \frac{\Delta m}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \]

   \[ \lim_{\Delta x \to 0} \left( \frac{\Delta m}{\Delta x} \right) = \frac{dm}{dx} = p \quad \text{the rate of change of mass with respect to length} \]

3. **Current**

   - Exists whenever electric charges move. If \( \Delta Q \) is the net charge passing through a surface during a time \( \Delta t \)

   \[ \text{Average current} \; I = \frac{\Delta Q}{\Delta t} = \frac{Q_2 - Q_1}{t_2 - t_1} \]
\[ I = \lim_{\Delta t \to 0} \frac{\Delta \Phi}{\Delta t} = \frac{d\Phi}{dt}. \]

* Current is the rate at which charge flows through a surface.

**Biology**

Let \( n = f(t) \) be the number of organisms in a plant or animal population.

Change in population between \( t = t_1 \) and \( t_2 \)

\[ \Delta n = f(t_2) - f(t_1) \]

so the average rate of growth in \( t_1 \leq t \leq t_2 \)

is \[ \frac{\Delta n}{\Delta t} \]

**Instantaneous rate of growth** \[ = \lim_{\Delta t \to 0} \frac{\Delta n}{\Delta t} = \frac{dn}{dt} \]

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**Economics**

If \( C(x) \) is the total cost of producing \( x \) units of a commodity.

If production is increased from \( x_1 \) to \( x_2 \),

average rate of change of cost \[ = \frac{\Delta C}{\Delta x} = \frac{C(x_2) - C(x_1)}{x_2 - x_1} \]

Suppose the population of bacteria doubles every hour.

\[ f(1) = 2f(0) = n_0 \]
\[ f(2) = 2f(1) = 2^2n_0 \]
\[ f(3) = 2 \cdot f(2) = 2^3n_0 \]
\[ \vdots \]
\[ f(t) = 2^t n_0 \]

\[ f'(t) = n_0 2^t \ln(2), \text{ rate of growth of bacteria} \]

If \( n_0 = 100 \), \[ \frac{dn}{dt} = 100 \cdot 2^t \ln(2) \text{ bacteria per hour.} \]
Marginal cost \( = \lim_{\Delta x \to 0} \frac{\Delta C}{\Delta x} = \frac{dC}{dx} \)

Taking \( \Delta x = 1 \), \( C'(n) \equiv C(n+1) - C(n) \),

\[ \text{Rate of increase of cost when production is } x. \]

\[ A + B \rightarrow C. \]

Let \( [C] \) be the concentration at time \( t \).

\[ \frac{\Delta [C]}{\Delta t} = \frac{[C](t_2) - [C](t_1)}{t_2 - t_1}, \]

\[ \text{Average rate of reaction} \]

Instantaneous rate of reaction \( \lim_{\Delta t \to 0} \frac{\Delta [C]}{\Delta t} = \frac{d[C]}{dt} \).

Examples

1. Blood flow through veins

\[ u(r) = \frac{P}{4\eta l} (R^2 - r^2) \]

\( \eta \) = viscosity

\( P \) = pressure difference in the tube

\( l \) = length of the tube

Instantaneous rate of change of velocity with respect to \( r \) = \( \frac{dv}{dr} = \frac{P}{4\eta l} (-2r) \).

\[ \text{blood flow is greatest along the center.} \]
3.7  Examples

1. The height (in metres) of a projectile shot vertically upwards from a point 2 m above the ground is

\[ h = 2 + 24.5t - 4.9t^2 \]  \quad \text{(distance in metres, time in s)}

(a) Find the velocity after 2 s, 4 s.
(b) When does the projectile reach its maximum height?
(c) What is the maximum height?
(d) When does it hit the ground?
(e) What is the velocity when it hits the ground?

\textbf{Solution}

(a) The velocity is \( \frac{dh}{dt} \) or \( h'(t) \)

\[ h'(t) = 24.5 - 9.8t \]

(i) \( h'(2) = 24.5 - 19.6 = 4.9 \text{ m/s} \)
(ii) \( h'(4) = 24.5 - 9.8(4) = 24.5 - 39.2 = -14.7 \text{ m/s} \)

(b) At the maximum height the projectile momentarily stops moving so

\[ h'(t) = 0 \]

So we find \( t \) such that

\[ 24.5 - 9.8t = 0 \]

\[ \frac{24.5}{9.8} = t \quad \Rightarrow \quad t = 2.5 \text{ s} \]

(c) Maximum height

\[ h(2.5) = 2 + 24.5(2.5) - 4.9(2.5)^2 \]

\[ = 28.326 \text{ m} \]

(d) Hit the ground when \( h(t) = 0 \)

\[ t = \frac{-24.5 \pm \sqrt{(24.5)^2 - 4(-4.9)(2)}}{2(-4.9)} \]

\[ \Rightarrow \quad t = 5.08 \quad \text{or} \quad -0.08 \]

Discard the negative root: \( t = -0.08 \)
2. The frequency of a vibrating string is given by

\[ f = \frac{1}{2L} \sqrt{\frac{T}{\rho}} \]

- \( L \) - length of the string
- \( T \) - tension
- \( \rho \) - linear density

(a)

Find the rate of change of frequency with respect to

(i) Length (L), (T and \( \rho \)) are constants.

(ii) the tension (T) (T and \( \rho \)) are constants

(iii) the linear density \( \rho \) (L and \( T \)) are constants.

**Solution**

(a) (i) \( f(L) = \frac{1}{2L} \sqrt{\frac{T}{\rho}} = \left( \frac{1}{2L} \right) \sqrt{\frac{T}{\rho}} \)

\[ \frac{df}{dL} = f'(L) \text{ is the rate of change of frequency w.r.t } L. \]

\[ f'(L) = -\frac{1}{2L} \sqrt{\frac{1}{\rho}} \cdot \frac{1}{L} = -\frac{1}{2L} \sqrt{\frac{T}{\rho}} \cdot \frac{1}{L^2} \]

(ii) \( f(T) = \frac{1}{2L} \sqrt{\frac{T}{\rho}} = \left( \frac{1}{2L} \right) \sqrt{T} = \left( \frac{1}{2L} \right) T^{1/2} \)

\[ f'(T) = \left( \frac{1}{2L} \right) \cdot \frac{1}{2} T^{-1/2} = \frac{1}{4L\sqrt{T}} \]

(iii) \( f(\rho) = \frac{1}{2L} \sqrt{T} \rho^{-1/2} \)

\[ f'(\rho) = \frac{1}{2L} \left( -\frac{1}{2} \right) \rho^{-3/2} \]