**Major Theme**

Development of numerical/computational techniques for solving various problems

e.g. 1. Approximate derivatives

2. Approximate integrals

3. Solutions of ODEs and PDEs.

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**Taylor's Theorem**

Let \( f, f', f'', \ldots, f^{(n)} \) be continuous on \([a, b]\) (i.e. \( f \in C^{n+1}[a, b]\)) and let \( f^{(n+1)}(x) \) exist for \( x \in (a, b) \). Then there exists a number \( c_x \in (a, x) \) such that

\[
f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots + \frac{(x-a)^n}{n!}f^{(n)}(a) + \frac{(x-a)^{n+1}}{(n+1)!}f^{(n+1)}(c_x)\]

In short,

\[
f(x) = \sum_{j=0}^{n} \frac{(x-a)^j}{j!}f^{(j)}(a) + \frac{(x-a)^{n+1}}{(n+1)!}f^{(n+1)}(c_x)
\]

\( (f(x) = P_n(x) + R_n(x) ) \)

\( P_n(x) \)

\( R_n(x) \)

Depending on \( f \), the Taylor series may converge to \( f \) everywhere or in some interval near \( x = a \).

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**Remarks**

1. \( P_n(x) \) - \( n \)-th degree Taylor polynomial of \( f \) at \( x = a \).

2. \( f(x) \) is the sum of its Taylor series if

\[
f(x) = \lim_{n \to \infty} T_n(x)\]

3. We will assume that \( f(x) \) has a Taylor polynomial representation so that for \( n \) large enough and \( x \) close to \( a \)

\[ f(x) \approx P_n(x) \]
1. Easy to evaluate

2. Weierstrass Approximation Theorem — uniformly

If \( f \) is continuous on \([a,b]\) then \( f \) can be approximated on that interval by polynomials to any degree of accuracy.

i.e. we can make \( |f(x) - P_n(x)| \) as small as we like!

**Poly**

*Let \( f \) be continuous on \([-1,1]\), and let \( \epsilon > 0 \) be arbitrary. Then there exists \( P \) s.t. \( \|f - P\| < \epsilon \)

**Polynomial approximate interpolation**

Interpolation — finding and evaluating a function that passes through a given set of data points.

**Lagrange Interpolating Polynomials**

**Simplest case — linear**

Find a function passing through \( \{(x_0, y_0), (x_1, y_1)\} \).

Let \( P_i(x) = c_0 + c_1 x_i \), so that

\[
P_i(x_0) = c_0 + c_1 x_0 = y_0 \quad \cdots (i)
\]

\[
P_i(x_1) = c_0 + c_1 x_1 = y_1 \quad \cdots (ii)
\]

Solve for \( c_0 \) and \( c_1 \)

\[
(ii) - (i) \Rightarrow c_1 (x_1 - x_0) = y_1 - y_0 \Rightarrow c_1 = \frac{y_1 - y_0}{x_1 - x_0}
\]

Plug \( c_1 \) into (i) \( \Rightarrow c_0 = y_0 - \left( \frac{y_1 - y_0}{x_1 - x_0} \right) x_0 \)

\[
= \frac{y_0 (x_1 - x_0) - x_0 (y_1 - y_0)}{x_1 - x_0}
= \frac{y_0 x_1 - y_0 x_0 - x_0 y_1 + x_0 y_0}{x_1 - x_0} = \frac{y_0 x_1 - x_0 y_1}{x_1 - x_0}
\]

\[
P_i(x) = \left( \frac{y_0 x_1 - x_0 y_1}{x_1 - x_0} \right) \frac{x}{x - x_0} + \left( \frac{y_1 - y_0}{x_1 - x_0} \right) x
\]
ranging

\[ P_i(x) = \left( \frac{x_i y_0 - y_0 x_i}{x_i - x_0} \right) + \left( \frac{y_1 x - x_0 y_1}{x_1 - x_0} \right) \]

\[ = \left( \frac{x - x_0}{x_0 - x_i} \right) y_x + \left( \frac{x - x_0}{x_0 - x_1} \right) y_1 \]

\[ \Rightarrow P_0(x) y_0 + P_i(x) y_1 = P_i(x) \]

Where

\[ P_0(x) = \frac{(x - x_1)}{(x_0 - x_1)} \quad \text{and} \quad P_i(x) = \frac{(x - x_i)}{(x_1 - x_i)} \]

are called Lagrange polynomials.

Degree n interpolating polynomial

**Given** \((n+1)\) data points with distinct \(x_i\) (interpolating nodes), The Lagrange
interpolating polynomial of degree \(\leq n\) is

\[ P_n(x) = y_0 P_0(x) + y_1 P_1(x) + \ldots + y_n P_n(x) \]

\[ \left( \sum_{l=0}^{n} y_l \phi_l(x) \right) \]

where

\[ \phi_l(x) = \frac{(x - x_0)(x - x_1) \cdots (x - x_{l-1})(x - x_{l+1}) \cdots (x - x_n)}{(x_l - x_0)(x_l - x_1) \cdots (x_l - x_{l-1})(x_l - x_{l+1}) \cdots (x_l - x_n)} \]

\[ = \prod_{j=0}^{n} \frac{(x - x_j)}{(x_l - x_j)} \quad \text{and} \quad \delta_{ij} = \begin{cases} 0, & \text{if } i \neq j, \\ 1, & \text{if } i = j. \end{cases} \]

Kronecker delta function.

**Example**

Given \(\{(1,2), (2,3)\}\) and \(\{(3,6)\}\), then the Lagrange interpolating polynomial is

\[ P_2(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} y_0 + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} y_1 + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} y_2 \]

\[ = \frac{(x-2)(x-3)}{1(3)} 2 + \frac{(x-1)(x-2)}{2(3)} 3 + \frac{(x-1)(x-2)}{2(-1)} 6 = (x-2)(x-3) + (x-1)(x-2) + (x-1)(x-2) \]
Theorem

Let \( x_0, x_1, x_2, \ldots, x_n \) be \((n+1)\) distinct points on \([a,b]\) and \( f \) be continuous on \([a,b]\) with \((n+1)\) continuous derivatives on \((a,b)\), then for each \( x \) on \([a,b]\)

there exists \( c_x \in [a,b] \) such that

\[
f(x) = P_n(x) + \frac{f^{(n+1)}(c_x)}{(n+1)!} (x-x_0)(x-x_1)(x-x_2) \cdots (x-x_n)
\]

where \( P_n(x) \) is the interpolating polynomial.

Notes

1. \( f(x) = P_n(x) \Rightarrow \) error is zero at interpolating nodes by construction.