This exam contains 6 pages (including this cover page) and 4 problems including an optional 5 point bonus question. Check to see if any pages are missing. Enter all requested information on the top of this page.

You may not use your books, notes, or any calculator on this exam.

To receive full credit you must provide all details. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.

- **Mysterious or unsupported answers will not receive any credit.** A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

- If you need more space, use the back of the pages; clearly indicate when you have done this.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Points</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>30</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>30</td>
<td></td>
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<tr>
<td>3</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>30</td>
<td></td>
</tr>
<tr>
<td><strong>Total:</strong></td>
<td><strong>105</strong></td>
<td></td>
</tr>
</tbody>
</table>
1. Existence and uniqueness theory

(a) (20 points) Discuss the existence and uniqueness of solutions for each of the following initial value problems. Please do not attempt to solve the initial value problems, use the theory discussed in class to determine the existence and uniqueness of a solution.

1. \( y'(t) = \tan(t), \quad y(0) = 1. \)

\[ f(t,y) = \tan(t) \text{ is continuous around } (0,1). \text{ However, } \]
\[ \frac{df}{dy} = \tan(t) \text{ is continuous for } -\frac{\pi}{2} < t < \frac{\pi}{2}, \text{ therefore } \]
there exists a local unique solution.

2. \( y'(t) = \frac{3}{2} y^{\frac{1}{3}}, \quad y(0) = 0. \)

\[ f(t,y) = \frac{3}{2} y^{\frac{1}{3}} \text{ is discontinuous at } (0,0). \text{ Therefore, the } \]
problem does not have a unique solution.

3. \( y'(t) = \sin(y), \quad y(0) = 0. \)

\[ f(t,y) = \sin(y) \text{ is continuous for all } (t,y). \]
\[ \frac{df}{dy} = \cos(y) \text{ is uniformly bounded therefore the } \]
problem has a unique global solution.

(b) (10 points) Suppose you have been hired as a numerical analyst, a colleague approaches you with an initial value problem describing the growth of the population of a bacterial colony.

1. What theoretical conditions would you check on the initial value problem to ensure that you can apply the numerical techniques we have discussed with confidence.

   For \( y = f(t,y), y(0) = y_0, \) \( f \) must be continuous and \( \frac{df}{dy} \) must be continuous in some region containing the initial condition.

   Further, we also need to ensure that the problem is stable: small changes in the initial condition yield small changes in the solution.

2. Which method would you use? Why?

   RK4 is most efficient.
2. Euler's method - it's ancient but it works!

(a) (5 points) State Euler's method for solving an initial value problem

\[ y'(t) = f(t, y(t)), \quad y(0) = y_0, \quad 0 \leq t \leq 1 \]

\[ y_{k+1} = y_k + hf(t_k, y_k), \quad k = 0, 1, 2, \ldots \]

(b) (10 points) Prove that Euler's method is consistent.

From Euler's theorem we have

\[ y(t_{kn}) = y(t_k) + h y'(t_k) + \frac{h^2}{2} y''(c_k), \quad c_k \in (t_k, t_{kn}) \]

Then recalling that \( y'(t) = f(t, y(t)) \)

\[ T_k = \frac{y_{k+1} - y_k}{-f(t_k, y_k)} = \frac{h}{2} y''(c_k) \]

So Euler's method is consistent and first order.

(c) (10 points) Suppose Euler's method is applied to the problem

\[ y'(t) = y(t), \quad y(0) = 1 \]

with a true solution of \( y(t) = e^t \). Show that the solution of Euler's method at each time step can be written as

\[ y_n = (1 + h)^{t_n}, \quad n \geq 0 \]

where \( t_n = nh \) and \( h \) is the size of the time step.

Hint: first show that \( y_n = (1 + h)^n \).

\[ y_1 = y_0 + hf(t_0, y_0) = 1 + h \]

\[ y_2 = y_1 + hf(t_1, y_1) = y_1 + h(1+h) = (1+h)^2 \]

Assuming that \( y_{n-1} = (1+h)^{n-1} \), then applying 1 step of Euler yields

\[ y_n = y_{n-1} + hf(t_{n-1}, y_{n-1}) = (1+h)^{n-1} + h \left[ (1+h)^{n-1} \right] = (1+h)(1+h)^{n-1} = (1+h)^n \]

Recalling that \( h = \frac{t_n}{n} \), yields the desired result.
(d) (5 points) **BONUS - A blast from the (Calc I) past!**
Prove that as $h$ decays the solution of Euler’s method at $t_n = 1$ converges to the true
solution by using *L’Hospital’s rule* to demonstrate that
\[
\lim_{h \to 0} (1 + h)\frac{1}{h} = e
\]

*Hint: the natural log is your friend.*

\[
y = (1 + h)^{\frac{1}{h}} \Rightarrow \ln(y) = \frac{1}{h} \ln(1 + h)
\]

\[
\lim_{h \to 0} \frac{\ln(1 + h)}{h} \overset{\text{H}}{=} \lim_{h \to 0} \left( \frac{1}{1 + h} \right) = 1
\]

So \( y = e^1 \) or \( y \to 0 \)

3. **High order methods**
The graph below is a plot of the logarithm of the global error over time for *Heun’s method*,
Euler’s method and the fourth order *Runge-Kutta method* for various values of $h$ to
ensure the same computational effort.

(a) (5 points) Explain the trend in the global error.

At each step the numerical schemes commit an error of the size $O(h^p)$ be the local truncation error.

This error accumulates as $t \to \infty$.

The higher the order of the method, the smaller the error at each step.
(b) (5 points) Identify methods A, B and C. Explain your reasoning.

B - Euler \( O(h) \)
A - RK2 \( O(h^2) \)
C - RK4 \( O(h^4) \)

(c) (5 points) Suppose Euler’s method is run with time step size \( h_0 \). How large would the time steps for RK4 and Heun’s method need to be to ensure a fair comparison as shown given figure? Explain.

\[
\begin{align*}
  h_0 &= \text{Euler} \\
  2h_0 &= \text{RK2} \\
  4h_0 &= \text{RK4}
\end{align*}
\]

To ensure the same number of function evaluations pick proportioned values \( h_0 \) for RK2 and RK4

4. Stability – Absolutely!

(a) (5 points) Define the term stiff problem

Coupled problems with components that evolve at very different time scales

\textbf{OR}

In the linear case, a system with eigenvalues with a large negative real part.

(b) (5 points) For the initial value problem

\[
y'(t) = \lambda y(t), \quad y(0) = y_0, \quad \text{Real}(\lambda) < 0 \quad (p)
\]

Define the region of absolute stability of a method.

The set of all values \( hL \in \mathbb{C} \) such that \( y_k \rightarrow 0 \) as \( k \rightarrow \infty \) when a numerical method is applied to \((p)\).
(c) (15 points) Show that the region of absolute stability of Euler’s method is a unit disk centred at \((-1,0)\).

(d) (5 points) Give an example of an A-stable method and state the advantage of this class of methods compared to other methods we have discussed this semester.

Backward Euler or Trapezoidal
These methods will yield a stable solution with no restriction on step size.