Homework #2

1. Let \( P_m(x) = a_0 + a_1 x + \ldots + a_m x^m \), then

\[
\ln P_m(x) = \ln (a_0 + a_1 x + \ldots + a_m x^m)
= \ln (a_0) + a_1 \ln (x) + \ldots + a_m \ln (x^m)
\]

(due to linearity of \( \ln \))

\[
\ln (x) = \sum_{i=1}^{m} w_i f(x_i)
\]

We are given that \( \ln \) integrates \( x^i \) exactly, therefore each term in the sum above is integrated exactly.

2. Use the method of undetermined coefficients using the values 0, \( \frac{1}{3} \), \( \frac{2}{3} \) and 1

Let find \( A_1, \ldots, A_4 \) so that

\[
\int_0^1 f(x) = A_1 f(0) + A_2 f\left(\frac{1}{3}\right) + A_3 f\left(\frac{2}{3}\right) + A_4 f(1)
\]

Integrates \( 1, x, x^2 \) and \( x^3 \) exactly, i.e.

\[
f(x) = 1 \quad A_1 + A_2 + A_3 + A_4 = 1 \quad \ldots (i)
\]

\[
f(x) = x \quad \frac{1}{3} A_2 + \frac{2}{3} A_3 + A_4 = \frac{1}{2} \quad \ldots (ii)
\]

\[
f(x) = x^2 \quad \frac{1}{9} A_2 + \frac{4}{9} A_3 + A_4 = \frac{1}{3} \quad \ldots (iii)
\]

\[
f(x) = x^3 \quad \frac{1}{27} A_2 + \frac{8}{27} A_3 + A_4 = \frac{1}{4} \quad \ldots (iv)
\]

Solving (however you like!) yields \( A_1 = A_4 = \frac{1}{8} \), \( A_2 = A_3 = \frac{3}{8} \), therefore the integration formula is

\[
\int_0^1 f(x) dx = \frac{1}{8} f(0) + \frac{3}{8} f\left(\frac{1}{3}\right) + \frac{3}{8} f\left(\frac{2}{3}\right) + \frac{1}{8} f(1)
\]

3. (a) \( a \) must satisfy:

\[
\int_{-1}^{1} dx = 2 \Rightarrow 1 + 1 = 2 \quad (\text{always true})
\]

\[
\int_{-1}^{1} x dx = 0 \Rightarrow a + (-a) = 0 \quad (\text{also, always true})
\]

This means that the formula is exact for \( 1, x \) independent of \( a \).
\( \alpha \) must satisfy
\[
\int_{-1}^{1} x^3 \, dx = \frac{2}{3} \quad \Rightarrow \quad 2 \alpha^3 = \frac{2}{3}
\]
\[
\int_{-1}^{1} x^3 \, dx = 0 \quad \Rightarrow \quad \alpha^3 + (-\alpha)^3 = 0 \quad \text{(always true)}
\]

The formula is exact for all polynomials of degree \( \leq 3 \) if and only if \( \alpha = \frac{\sqrt{3}}{3} \)

It follows from (a)(b) that the formula is exact integrating polynomials of the form \( a + bx + cx^3 \) exactly so we need to fix \( \alpha \) to integrate \( x^4 \) exactly. Indeed
\[
\int_{-1}^{1} x^4 \, dx = \frac{2}{5} \quad \Rightarrow \quad 2 \alpha^4 = \frac{2}{5} \quad \text{so} \quad \alpha = \left( \frac{1}{5} \right)^{\frac{1}{4}}
\]

4. (a)

![Graph](image)

\[
\int_{a}^{b} f(x) \, dx \approx \frac{h}{2} \sum_{i=1}^{n} f(x_i^{*}) \quad \quad x_i^{*} = \frac{1}{2} \left( x_{i-1} + x_{i+1} \right)
\]

(b) Let \( f(x) = a_0 + a_1 x \) on \([x_{i-1}, x_i] \), then
\[
\int_{x_{i-1}}^{x_i} f(x) \, dx = \int_{x_{i-1}}^{x_i} (a_0 + a_1 x) \, dx = a_0 x + a_1 \frac{x^2}{2} \bigg|_{x_{i-1}}^{x_i}
\]
\[
= a_0 (x_i - x_{i-1}) + a_1 \left( \frac{x_i^2 - x_{i-1}^2}{2} \right)
\]
Now check the midpoint method on \([x_{i-1}, x_i] \).
\[
h f(x_i^{*}) = (x_i - x_{i-1}) \left[ a_0 + a_1 \left( \frac{x_i + x_{i-1}}{2} \right) \right]
\]
\[
= a_0 (x_i - x_{i-1}) + a_1 \left( \frac{x_i^2 - x_{i-1}^2}{2} \right)
\]
\[
= a_0 (x_i - x_{i-1}) + a_1 \left( \frac{x_i^2 - x_{i-1}^2}{2} \right) \quad \blacksquare
\]
This is another technique for analyzing the error. In class, we integrated the interpolant, here we deal with the Taylor polynomial.

\[ f(x) = f(x_{i-1}) + (x-x_{i-1}) f'(x_{i-1}) + \frac{(x-x_{i-1})^2}{2} f''(c_i), \quad c_i \in [x_{i-1}, x_i] \]

Integrate each term and quantify

\[
\int_{x_{i-1}}^{x_i} f(x) \, dx = (x_i - x_{i-1}) f(x_{i-1})
\]

\[
\int_{x_{i-1}}^{x_i} f(x) \, dx = f(x_{i-1}) (x_i - x_{i-1}) + \frac{f'(x_{i-1})}{2} \int_{x_{i-1}}^{x_i} (x-x_{i-1}) \, dx + \frac{f''(c_i)}{2} \int_{x_{i-1}}^{x_i} (x-x_{i-1})^2 \, dx
\]

\[
= f(x_{i-1}) (x_i - x_{i-1}) + \frac{f''(c_i)}{24} (x_i - x_{i-1})^3
\]

We conclude that

\[
\int_{x_{i-1}}^{x_i} f(x) \, dx = f(x_{i-1}) (x_i - x_{i-1}) + \frac{f''(c_i)}{24} (x_i - x_{i-1})^3
\]

Summing up the errors yields

\[
\frac{1}{24} \sum_{L=1}^{n} f''(c_L) (x_L - x_{L-1})^3 = \frac{h^3}{24} \sum_{L=1}^{n} f''(c_L)
\]

We can quantify \( \sum_{L=1}^{n} f''(c_L) \) as we did for trapezoidal by noting that \( f''(x) \) is continuous, therefore there exist \( c_1 \) and \( c_2 \) in \( [a,b] \) such that

\[
f(c_1) \leq f''(x) \leq f''(c_2).
\]

Therefore

\[
r f''(c_1) \leq \sum_{L=1}^{n} f''(c_L) \leq r f''(c_2)
\]

so that

\[
f''(c_1) \leq \frac{1}{r} \sum_{L=1}^{n} f''(c_L) \leq f''(c_2)
\]

Then by IOT, there exist \( \eta \in (a,b) \) such that

\[
f''(\eta) = \frac{1}{r} \sum_{L=1}^{n} f''(c_L)
\]

and the error term becomes

\[
\frac{h^3}{24} \sum_{L=1}^{n} f''(c_L) = \frac{h^3}{24} r f''(\eta) = \frac{h^2 (b-a)}{24} f''(\eta).
\]
\[ f(x) = \frac{4}{1+x^2} \quad \int_0^1 f(x)dx \]

The error term for Trapezoidal is

\[ \frac{(b-a)h^2}{12} f''(\eta) = \frac{(b-a)^3}{12n^2} f''(\eta) \]

\[ h = \frac{b-a}{n} \]

\[ \max_{0 \leq x \leq 1} |f''(x)| = 8 \quad \left( f''(x) \text{ is a continuous function on a closed interval} \right) \]

So select \( n \) so that

\[ \frac{(1-0)^3 \cdot 8}{12n^2} < 1.0 \times 10^{-5} \]

Solving yields \( n = 258.19 \) so pick \( n = 259 \).