Second order Taylor Method

- Taylor expansion of $y(t + h)$ about $y(t)$ yields

\[
y(t + h) = y(t) + hy'(t) + \frac{h^2}{2}y''(t) + \frac{h^3}{3!}y^{(3)}(c_i)
\]

\[
= y(t) + hf(t, y(t)) + \frac{h^2}{2} \left( \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y} f \right)(t, y(t)) + \mathcal{O}(h^3)
\]
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\]

- Then the second order Taylor method is:

\[
\frac{y_{k+1} - y_k}{h} = f(t_k, y_k) + \frac{h}{2} \left( \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y} \right)(t_k, y_k)
\]
Second order Runge-Kutta method

Develop a second order method of the form:

\[ \frac{y_{k+1} - y_k}{h} = a_1 f(t, y) + a_2 f(t + \alpha_2, y + \delta_2 f(t, y)) \]
Second order Runge-Kutta method

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- We need to solve for \( a_1, a_2, \alpha_2 \) and \( \delta_2 \)
Second order Runge-Kutta method

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- To do that, we compare to second order Taylor:

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\]

- So we have to expand \(a_2 f(t + \alpha_2, y + \delta_2 f(t, y))\)
Generalized Taylor (Calc 3 version)

Theorem

Let $f(t, y)$ and all first and second order partial derivatives be continuous then

$$f(t + \Delta t, y + \Delta y) = f(t, y) + \left( \Delta t \frac{\partial f}{\partial t}(t, y) + \Delta y \frac{\partial f}{\partial y}(t, y) \right) + R$$
Generalized Taylor (Calc 3 version)

Theorem

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f(t + \Delta t, y + \Delta y) = f(t, y) + \left( \Delta t \frac{\partial f}{\partial t}(t, y) + \Delta y \frac{\partial f}{\partial y}(t, y) \right) + \mathcal{R}
\]

therefore setting \( \Delta t = \alpha_2, \quad \Delta y = \delta_2 f(t, y) \) yeilds

\[
f(t + \alpha_2, y + \delta_2 f(t, y)) = f(t, y) + \alpha_2 \frac{\partial f}{\partial t}(t, y) + \delta_2 f(t, y) \frac{\partial f}{\partial y}(t, y) + \mathcal{R}
\]
Comparing apples to apples

- **RK2**

\[
\frac{y_{k+1} - y_k}{h} = a_1 f(t, y) + a_2 (f(t, y) + \alpha_2 \frac{\partial f}{\partial t}(t, y) + \delta_2 f(t, y) \frac{\partial f}{\partial y}(t, y) + R)
\]

\[
= (a_1 + a_2) f(t, y) + a_2 \alpha_2 \frac{\partial f}{\partial t}(t, y) + a_2 \delta_2 f(t, y) \frac{\partial f}{\partial y}(t, y) + a_2 R
\]

- **Second order Taylor**

\[
\frac{y_{k+1} - y_k}{h} = f(t, y) + \frac{h}{2} \frac{\partial f}{\partial t}(t, y) + \frac{h}{2} f(t, y) \frac{\partial f}{\partial y}(t, y) + R
\]

- Therefore we need

\[
a_1 + a_2 = 1, \quad a_2 \alpha_2 = \frac{h}{2}, \quad a_2 \delta_2 = \frac{h}{2}
\]

- Which has infinitely many solutions of the form \(a_1 + a_2 = 1\) and

\[
\alpha_2 = \delta_2 = \frac{h}{2a_2}
\]