1.4.1c

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

$$u(x,0) = f(x)$$

$$\frac{\partial u}{\partial x}(0) = 0 \quad , \quad u(L) = T$$

Equilibrium solution

$\frac{\partial u}{\partial t} = 0 \Rightarrow$ $u''(x) = 0$ so the general solutions are of the form $u(x) = c_1 x + c_2$.

Applying boundary conditions

(i) $\frac{du}{dx}(0) = 0 \Rightarrow c_1 = 0$, (ii) $u(L) = T \Rightarrow c_2 = T$

The equilibrium solution is $u(x) = T$.

1.4.1f $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + Q$, $\frac{Q}{K_0} = x^2$

$u(0) = T$ $\frac{du}{dx}(L) = 0$, $u(x,0) = f(x)$.

Equilibrium solutions

$\frac{\partial u}{\partial t} = 0$ and recalling that $k = \frac{K_0}{\rho c}$ so that

$$\rho c \frac{du}{dt} = K_0 \frac{\partial^2 u}{\partial x^2} + Q$$, then the steady problem is

$$0 = K_0 \frac{d^2 u}{dx^2} + K_0 x^2 \Rightarrow \frac{d^2 u}{dx^2} = -x^2.$$

Solving $u''(x) = -x^2$ yields a general solution

$$u(x) = \frac{-x^4}{12} + c_1 x + c_2$$

Applying boundary conditions

$u(0) = T \Rightarrow c_2 = T$

$\frac{du}{dx} = -\frac{x^3}{3} + c_1$, $\frac{du}{dx}(L) = 0 \Rightarrow -\frac{L^3}{3} + c_1 = 0 \Rightarrow c_1 = \frac{L^3}{3}$

$$u(x) = \frac{-x^4}{12} + \frac{L^3}{3} x + T$$

1.4.1 g

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

$$u(0) = T, \qquad \frac{\partial u}{\partial x}(L) + u(L) = 0.$$

Equilibrium solution

Indeed, $u''(x) = 0 \Rightarrow u(x) = c_1 x + c_2.$

Boundary conditions

$u(0) = T \Rightarrow c_2 = T.$

$\frac{\partial u}{\partial x}(L) + u(L) = 0 \Rightarrow c_1 + (c_1 L + T) = 0 \Rightarrow c_1(1+L) = -T$

$c_1 = \frac{-T}{1+L}$

so $u(x) = \left(\frac{-T}{1+L}\right)x + T.$

1.4.2.

(a) Heat energy generated per unit time inside rod $= \int_0^L Q\, A\,dx \underset{Q = K_0 x}{=} A K_0 \int_0^L x\,dx = A K_0 \left.\frac{x^2}{2}\right|_0^L = \frac{A K_0 L^2}{2}.$

(b) Heat energy flowing out per unit time @ $x = 0$ and $x = L$

From FOURIER's Law,

$$\phi(x,t) = -K_0 \frac{\partial u}{\partial x} \qquad (*)$$

The heat rod is in equilibrium with $\frac{Q}{K_0} = x$ so we can solve for the equilibrium temperature distribution, $u(x)$ then compute $\phi(x,t)$.

Equilibrium solution

$$\rho c \frac{\partial u}{\partial t} = K_0 \frac{\partial^2 u}{\partial x^2} + Q \underset{\frac{\partial u}{\partial t} = 0}{\Rightarrow} 0 = K_0 \frac{d^2 u}{dx^2} + x K_0$$

$$\Rightarrow u''(x) = -x$$

Solving yields $$u(x) = \frac{-x^3}{6} + c_1 x + c_2$$

Applying boundary conditions:

$$u(0) = u(L) = 0$$

$$u(0) = 0 \Rightarrow c_2 = 0$$

$$u(L) = 0 \Rightarrow -\frac{L^3}{6} + c_1 L = 0 \Rightarrow c_1 L = \frac{L^3}{6} \Rightarrow c_1 = \frac{L^2}{6}$$

So $$u(x) = -\frac{x^3}{6} + \frac{L^2}{6}x \Rightarrow \frac{du}{dx} = -\frac{1}{2}x^2 + \frac{L^2}{6}$$

so $\phi(x) = -K_0 \frac{du}{dx}$ so the heat flowing out of the rod per unit time

@ $\underline{x=0}$

$$A\phi(0) = A\left(-K_0\left(-\frac{1}{2}\cdot 0 + \frac{L^2}{6}\right)\right) = \frac{-AK_0L^2}{6}$$

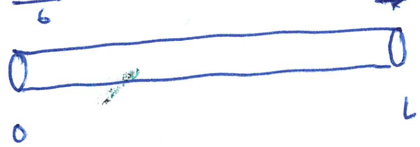
@ $x=L$

$$A\phi(L) = -AK_0\left(-\frac{1}{2}L^2 + \frac{L^2}{6}\right) = \frac{AK_0L^2}{3}$$

(c) The heat rod is in equilibrium

$\phi = -\frac{AK_0L^2}{6} < 0$ $\leftarrow$ heat flowing out to the left.

$\rightarrow \phi = \frac{AK_0L^2}{3} > 0$ (heat flowing out to the right)



The sum in absolute value of heat flowing out at $x=0$ and $x=L$ is equal to the amount of heat generated in the rod.

1.4.4. $$c\rho \frac{\partial u}{\partial t} = K_0 \frac{\partial^2 u}{\partial x^2}$$

Integrating on both sides

$$c\rho \int_0^L \frac{\partial u}{\partial t}\, dx = K_0 \int_0^L \frac{\partial^2 u}{\partial x^2} \Rightarrow c\rho \frac{d}{dt} \int_0^L u\, dx = K_0 \frac{\partial u}{\partial x}\Big|_0^L$$

$$= K_0 \frac{\partial u}{\partial x}(L) - K_0 \frac{\partial u}{\partial x}(0)$$

The rod is insulated, therefore we have $K_0 \frac{\partial u}{\partial x}(L) - K_0 \frac{\partial u}{\partial x}(0) = 0$

We have established that

$$\frac{d}{dt} \int_0^L c\rho u\, dx = 0 \Rightarrow \int_0^L c\rho u\, dx \text{ is a constant in time.}$$

This proves that the thermal energy is constant because

$$E(t) = \int_0^L e(x,t)\, A\, dx = A \int_0^L c\rho u\, dx.$$

1.4.7b.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad u(x,0) = f(x) \quad \frac{\partial u}{\partial x}(0,t) = 1, \quad \frac{\partial u}{\partial x}(L,t) = \beta.$$

Equilibrium solution

$$u''(x) = 0 \Rightarrow u(x) = c_1 x + c_2$$

Applying boundary conditions, we have

$\frac{du}{dx} = c_1$ so $\frac{du}{dx}(0) = 1 \Rightarrow c_1 = 1$ $\quad \frac{du}{dx}(L) = \beta = c_1 = 1$ for a solution to exist.

At this point c_2 is free but we know that the total energy at the start should equal the total energy at equilibrium

(dividing out constants)

so
$$\int_0^L f(x)\,dx = \int_0^L x + c_2 \, dx$$
$$= \frac{x^2}{2} + c_2 x \Big|_0^L = \frac{L^2}{2} + c_2 L$$

solving
$$c_2 = \frac{1}{L}\left(\int_0^L f(x)\,dx - \frac{L^2}{2}\right)$$

So
$$u(x) = x + \frac{1}{L}\left(\int_0^L f(x)\,dx - \frac{L^2}{2}\right)$$

1.4.10. $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 4$, $u(x,0) = f(x)$, $\frac{\partial u}{\partial x}(0,t) = 5$, $\frac{\partial u}{\partial x}(L,t) = 6.$

Integrating both sides of the Pde yeilds
$$\int_0^L \frac{\partial u}{\partial t}\,dx = \int_0^L \frac{\partial^2 u}{\partial x^2} + \int_0^L 4\,dx$$

so that
$$\frac{d}{dt}\int_0^L u\,dx = \frac{\partial u}{\partial x}\Big|_0^L + 4x\Big|_0^L$$
$$\frac{d}{dt}\int_0^L u\,dx = \frac{\partial u}{\partial x}(L,t) - \frac{\partial u}{\partial x}(0,t) + 4L$$
$$= (6-5) + 4L$$
$$= 4L+1.$$

Integrating in t then gives
$$\int_0^L u\,dx = \int 4L+1\,dt$$
$$= (4L+1)t + C$$

So we have ~~the~~ $E(t) = (4L+1)t + C$, $E(0) = C$

and $E(0)$ is the initial energy $= \int_0^L f(x)\,dx$ so $E(t) = (4L+1)t + \int_0^L f(x)\,dx.$