\[
\frac{du}{dt} = k \frac{d^2u}{dx^2} \\
u(x,0) = f(x) \\
\frac{du}{dx}(0) = 0, \quad u(L) = T
\]

**Equilibrium solution**

\[
\frac{du}{dt} = 0 \Rightarrow u''(x) = 0 \quad \text{so the general solution are of the form} \quad u(x) = C_1 x + C_2.
\]

Applying boundary conditions

(i) \( \frac{du}{dx}(0) = 0 \Rightarrow C_1 = 0 \), \quad (ii) \( u(L) = T \Rightarrow C_2 = T \)

The equilibrium solution is \( u(x) = T \).

1. If \( \frac{du}{dt} = k \frac{d^2u}{dx^2} + Q \), \( \frac{Q}{K_0} = x^2 \)

\( u(0) = T \), \( \frac{du}{dx}(L) = 0 \), \( u(x,0) = f(x) \).

**Equilibrium solutions**

\[
\frac{du}{dt} = 0 \quad \text{and recalling that} \quad k = \frac{K_0}{pc} \quad \text{so that}
\]

\[
pc \frac{du}{dt} = K_0 \frac{d^2u}{dx^2} + Q, \quad \text{then the steady problem is}
\]

\[
0 = K_0 \frac{d^2u}{dx^2} + K_0 x^2 \quad \Rightarrow \quad \frac{d^2u}{dx^2} = -x^2.
\]

Solving \( u''(x) = -x^2 \) yields a general solution

\[
u(x) = -\frac{2x^4}{12} + C_1 x + C_2
\]

Applying boundary conditions

\( u(0) = T \Rightarrow C_2 = T \)

\[
\frac{du}{dx} = -\frac{x^3}{3} + C_1, \quad \frac{du}{dx}(L) = 0 \Rightarrow -\frac{L^3}{3} + C_1 = 0 \Rightarrow C_1 = \frac{L^3}{3}
\]

\[
u(x) = -\frac{2x^4}{12} + \frac{L^3}{3} x + T
\]
\[ \frac{du}{dt} = k \frac{d^2u}{dx^2} \]

\[ u(0) = T', \quad \frac{du}{dx} (L) + u(L) = 0. \]

**Equilibrium solution**

Indeed, \( u''(x) = 0 \) \( \Rightarrow \) \( u(x) = C_1 x + C_2. \)

**Boundary conditions**

\[ u(0) = T \quad \Rightarrow \quad C_2 = T. \]

\[ \frac{du}{dx} (L) + u(L) = 0 \quad \Rightarrow \quad C_1 + (C_1 L + T) = 0 \quad \Rightarrow \quad C_1 (1 + L) = -T \]

\[ C_1 = \frac{-T}{1+L} \]

So \( u(x) = \left( \frac{-T}{1+L} \right) x + T. \)

1.4.2.

(a) Heat energy generated per unit time inside rod

\[ \int_0^L Q \, Adx = A k_0 \int_0^L x \, dx = A k_0 \frac{x^2}{2} \bigg|_0^L \]

\[ Q = k_0 x \]

(b) Heat energy flowing out per unit time at \( x = 0 \) and \( x = L \)

From Fourier's law,

\[ \phi(x,t) = -k_0 \frac{du}{dx} \quad \text{(*)} \]

The heat rod is in equilibrium \( \frac{Q}{k_0} = x \) so we can solve for the equilibrium temperature distribution \( u(x) \) then compute \( \phi(x, t) \).

**Equilibrium solution**

\[ \alpha e \frac{du}{dt} = k_0 \frac{d^2u}{dx^2} + Q \quad \Rightarrow \quad 0 = k_0 \frac{d^2u}{dx^2} + 2k_0 \]

\[ \frac{du}{dt} = 0 \quad \Rightarrow \quad u''(x) = -x \]
Solving yields \( u(x) = \frac{-x^3}{6} + c_1 x + c_2 \)

Applying boundary conditions:

\[ u(0) = u(L) = 0 \]

\[ u(0) = 0 \Rightarrow c_2 = 0 \]

\[ u(L) = 0 \Rightarrow -\frac{L^3}{6} + c_1 L = 0 \Rightarrow c_1 L = \frac{L^3}{6} \Rightarrow c_1 = \frac{L^2}{6} \]

So

\[ u(x) = \frac{-x^3}{6} + \frac{L^2}{6} x \Rightarrow \frac{du}{dx} = -\frac{1}{2} x^2 + \frac{L^2}{6} \]

So

\[ \phi(x) = -K_0 \frac{du}{dx} \Rightarrow \text{the heat flowing out of the rod per unit time} \]

\[ \text{at } x=0 \]

\[ A \phi(0) = A \left( -K_0 \left( -\frac{1}{2} x^2 + \frac{L^2}{6} \right) \right) = -\frac{AK_0 L^2}{6} \]

\[ \text{at } x=L \]

\[ A \phi(L) = -AK_0 \left( -\frac{1}{2} L^2 + \frac{L^2}{6} \right) = \frac{AK_0 L^2}{3} \]

\[ \text{The heat rod is in equilibrium} \]

\[ \phi = -\frac{AK_0 L^2}{6} < 0 \Rightarrow \phi = \frac{AK_0 L^2}{3} > 0 \text{ (heat flowing out to the right)} \]

\[ \text{Heat flowing out to the left.} \]

The sum in absolute value of heat flowing out at \( x=0 \) and \( x=L \) is equal to the amount of heat generated in the rod.
1.4.4. \[ c_s \frac{du}{dt} = K_0 \frac{d^2 u}{dx^2} \]

Integrating on both sides:
\[ c_s \int_0^L \frac{du}{dt} \, dx = K_0 \int_0^L \frac{d^2 u}{dx^2} \, dx \Rightarrow c_s \frac{d}{dt} \int_0^L u \, dx = K_0 \frac{du}{dx} \bigg|_0^L \]
\[ = K_0 \frac{du}{dx} (L) - K_0 \frac{du}{dx} (0) \]

The rod is insulated, therefore we have \( K_0 \frac{du}{dx} (L) = K_0 \frac{du}{dx} (0) \)

We have established that:
\[ \frac{d}{dt} \int_0^L c_s u \, dx = 0 \Rightarrow \int_0^L c_s u \, dx \text{ is a constant in time}. \]

This proves that the thermal energy is constant because:
\[ E(t) = \int_0^L \varepsilon(x,t) A \, dx = A \int_0^L c_s u \, dx. \]

1.4.7b.
\[ \frac{du}{dt} = \frac{d^2 u}{dx^2} \quad , \quad u(x,0) = f(x) \quad , \quad \frac{du}{dx} (0,t) = 1 \quad , \quad \frac{du}{dx} (L,t) = \beta \]

**Equilibrium Solution**
\[ u''(x) = 0 \Rightarrow u(x) = c_1 x + c_2 \]

Applying boundary conditions, we have:
\[ \frac{du}{dx} = c_1 \quad \text{so} \quad \frac{du}{dx} (0) = 1 \Rightarrow c_1 = 1 \]
\[ \frac{du}{dx} (L) = \beta = c_1 = 1 \quad \text{for a solution to exist}. \]

At this point \( c_2 \) is free, but we know that the total energy at the start should equal the total energy at equilibrium.
(dropping constants)

\[ \int_0^L f(x) \, dx = \left[ x + c_2 \right]_0^L = \frac{x^2}{2} + c_2 \cdot L \]

Solving \( c_2 = \frac{1}{L} \left( \int_0^L f(x) \, dx - \frac{L^2}{2} \right) \)

So \( u(x) = x + \frac{1}{L} \left( \int_0^L f(x) \, dx - \frac{L^2}{2} \right) \)

14.10. \( \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 4 \), \( u(x,0) = f(x) \), \( \frac{\partial u}{\partial x} (0,t) = 5 \), \( \frac{\partial u}{\partial x} (L,t) = 6 \).

Integrating both sides of the PDE yields

\[ \int_0^L \frac{\partial u}{\partial t} \, dx = \int_0^L \frac{\partial^2 u}{\partial x^2} \, dx + \int_0^L 4 \, dx \]

So that

\[ \frac{d}{dt} \int_0^L u \, dx = \left. \frac{\partial u}{\partial x} \right|_0^L + 4x \]

\[ \frac{d}{dt} \int_0^L u \, dx = \frac{\partial u}{\partial x} (L,t) - \frac{\partial u}{\partial x} (0,t) + 4L \]

\[ = (6 - 5) + 4L \]

\[ = 4L + 1 \]

Integrating in \( t \) then gives

\[ \int_0^L u \, dx = \int_0^L 4L + 1 \, dt \]

\[ = (4L + 1) \cdot t + C \]

So we have \( E(t) = (4L + 1) \cdot t + C \), \( E(0) = C \)

and \( E(0) \) is the initial energy = \( \int_0^L f(x) \, dx \) so \( E(t) = (4L + 1) \cdot t + \int_0^L f(x) \, dx \).