2.2.1

Show that any linear combination of Linear operators is a linear operator

PROOF

Let $\{L_1, L_2, \dots, L_n\}$ be a set of linear operators and let $c_i$ be scalars.

then

$$(c_1L_1 + c_2L_2 + c_3L_3 + \dots + c_nL_n)(u+v)$$

$$= c_1L_1(u+v) + c_2L_2(u+v) + \dots + c_nL_n(u+v)$$

$$= c_1L_1(u) + c_2L_1(v) + c_2L_2(u) + c_2L_2(v) + \dots + c_nL_n(u) + c_nL_n(v)$$

$$= (c_1L_1 + c_2L_2 + \dots + c_nL_n)(u) + (c_1L_1 + c_2L_2 + \dots + c_nL_n)(v)$$

Further, if $\alpha$ is a scalar

$$(c_1L_1 + c_2L_2 + \dots + c_nL_n)(\alpha u)$$

$$= (c_1L_1)(\alpha u) + (c_2L_2)(\alpha u) + \dots + (c_nL_n)(\alpha u)$$

$$= \alpha c_1 L(u) + \alpha c_2 L(u) + \dots + \alpha c_n L(u)$$

$$= \alpha(c_1L_1 + c_2L_2 + \dots + c_nL_n)(u)$$

$\square$

2.2.2 $L(u) = \frac{\partial}{\partial x}\left[K_0(x)\frac{\partial u}{\partial x}\right]$ is a linear operator.

$$L(u) = K_0'(x)\frac{\partial u}{\partial x} + K_0(x)\frac{\partial^2 u}{\partial x^2}$$ (assuming $u(x)$ is twice differentiable)

$$\cancel{L(c_1u_1 + c_2u_2) = K_0'(x)\, c_1\frac{\partial u_1}{\partial x} + K_0(x)\, c_2\frac{\partial^2 u_2}{\partial x^2}}$$

$$L(c_1u_1 + c_2u_2) = K_0'(x)\frac{\partial}{\partial x}(c_1u_1 + c_2u_2) + K_0(x)\frac{\partial^2}{\partial x^2}(c_1u_1 + c_2u_2)$$

$$= K_0'(x)\left(c_1\frac{\partial u_1}{\partial x} + c_2\frac{\partial u_2}{\partial x}\right) + K_0(x)\left(c_1\frac{\partial^2 u_1}{\partial x^2} + c_2\frac{\partial^2 u_2}{\partial x^2}\right)$$

$$= \left(c_1K_0'(x)\frac{\partial u_1}{\partial x} + c_1K_0(x)\frac{\partial^2 u_1}{\partial x^2}\right) + c_2K_0'(x)\frac{\partial u_2}{\partial x} + c_2K_0(x)\frac{\partial^2 u_2}{\partial x^2}$$

$$= c_1 L(u_1) + c_2 L(u_2) \qquad \square$$

(b) Pick any function that depends on $u$ e.g $K_0(x,u) = u$ and show that

$$L(c_1 u_1 + c_2 u_2) \neq c_1 L(u_1) + c_2 L(u_2)$$

2.2.3. $\dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2} + Q(u,x,t)$ is linear if $Q(u,x,t) = \alpha(x,t)u + \beta(x,t)$

$$\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} - \alpha(x,t)u = \beta(x,t)$$

In this case $L(u) = \dfrac{\partial u}{\partial t} - k\dfrac{\partial^2 u}{\partial x^2} - \alpha(x,t)u$.

You can show that $L(u)$ is a linear operator as in 2.2.2.

To prove ~~homo~~ that the PDE is homogeneous if $\beta(x,t) = 0$ we need to show that $u=0$ satisfies the pde.

Indeed

Let $u = 0$, then

$$L(u) = 0 \Rightarrow \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + \alpha(x,t)u$$

$$0 = k(0) + \alpha(x,t)\,0$$

$$0 = 0 \qquad \square.$$

2.3.3 (a)

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \qquad (**)$$

$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0$$

$$u(x,0) = 6\sin\left(\frac{9\pi}{L}x\right).$$

From separation of variables $u(x,t) = \phi(x)\,G(t)$ we have 2 odes

$$G'(t) = -\lambda k G \quad \text{and} \quad \frac{d^2\phi}{dx^2} = -\lambda\phi \qquad (*)$$

$$\phi(0) = \phi(L) = 0.$$

From the table the eigenvalue problem $*$ has eigenvalues

$$\lambda = \left(\frac{n\pi}{L}\right)^2 \quad \text{and eigenfunctions} \quad \phi(x) = c_1\sin\left(\frac{n\pi}{L}x\right), \quad n = 1,2,3,\ldots$$

The time dependent problem has solution

$$G(t) = c_2 e^{-\lambda k t} = c_2 e^{-\left(\frac{n\pi}{L}\right)^2 kt}$$

The product solution thus takes the form

$$u(x,t) = c\,e^{-k\left(\frac{n\pi}{L}\right)^2 t}\sin\left(\frac{n\pi}{L}x\right), \quad n = 1,2,3,\ldots$$

By the principle of superposition

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-k\left(\frac{n\pi}{L}\right)^2 t}\sin\left(\frac{n\pi}{L}x\right)$$ is the general

solution to $(**)$.

so $$u(x,0) = \sum_{n=1}^{\infty} c_n e^{-k\left(\frac{n\pi}{L}\right)^2\cdot 0}\sin\left(\frac{n\pi}{L}x\right) = \sum_{n=1}^{\infty} c_n\sin\left(\frac{n\pi}{L}x\right)$$

and for
$$\sum_{n=1}^{\infty} c_n\sin\left(\frac{n\pi}{L}x\right) = 6\sin\left(\frac{9\pi}{L}x\right),$$ we need

$c_n = 0 \ \forall n \neq 9$ and $c_9 = 6$ so that

$$\boxed{u(x,t) = 6e^{-k\left(\frac{9\pi}{L}\right)^2 t}\sin\left(\frac{9\pi x}{L}\right)}$$

2.3.3(b)

Follow closely the argument in 2.3.3a for each term in the initial condition.

$$u(x,0) = 3\sin\left(\frac{\pi}{L}x\right) - \sin\left(\frac{3\pi x}{L}\right)$$

You solution should be

$$u(x,t) = 3e^{-k\left(\frac{\pi}{L}\right)^2 t}\sin\left(\frac{\pi x}{L}\right) - e^{-k\left(\frac{3\pi}{L}\right)^2 t}\sin\left(\frac{3\pi x}{L}\right)$$

2.3.8(a)

Equilibrium solution

$$\frac{\partial u}{\partial t} = 0 \Rightarrow \boxed{k\frac{d^2u}{dx^2} = \alpha u}, \quad \alpha > 0.$$

The general solution is $u(x) = a\cosh\sqrt{\frac{\alpha}{k}}\,x + b\sinh\sqrt{\frac{\alpha}{k}}\,x$

Applying boundary conditions

$u(0) = 0 \Rightarrow a = 0$

$u(L) = 0 \Rightarrow b = 0$ so the equilibrium solution is 0.

2.3.8(b) Solving the time dependent problem

$$u = \phi(x)h(t) \Rightarrow \phi(x)\frac{dh}{dt} + \alpha\,\phi(x)h(t) = kh\frac{d^2\phi}{dx^2}$$

dividing by $k\phi h$ yeilds

$$\frac{1}{kh}\frac{dh}{dt} + \frac{\alpha}{k} = \frac{1}{\phi}\frac{d^2\phi}{dx^2} = -\lambda.$$

So we have

$$\frac{1}{kh}\frac{dh}{dt} + \frac{\alpha}{k} = -\lambda \quad \text{and} \quad \frac{d^2\phi}{dx^2} = -\lambda\phi$$

$$\phi(0) = \phi(L) = 0$$

$$\Downarrow \qquad\qquad \Downarrow \text{ * From table*}$$

$$\frac{h'(t)}{h(t)} + \alpha = -\lambda k$$

This, we know has eigenvalues $\lambda = \left(\frac{n\pi}{L}\right)^2$ with eigenfunctions ~~eigenvalues~~

$$\phi(x) = \sin\left(\frac{n\pi x}{L}\right)$$

Recall that $\frac{h'(t)}{h(t)} = \frac{d}{dt}\left(\ln(h(t))\right)$

$$\frac{d}{dt}\left(\ln(h(t))\right) = -\alpha - \lambda k$$

$$\ln(h(t)) = (-\alpha - \lambda k)\,t \Rightarrow h(t) = e^{-\lambda k t}\cdot e^{-\alpha t}$$

By superposition

$$u(x,t) = e^{-\alpha t}\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t} \quad \text{with} \quad b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

as $t \to \infty$, $u(x,t) \to 0$!