Homework #3 Partial solutions

1.
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0<x<L, \quad 0<y<H$$

$$\frac{\partial u}{\partial x}(0,y) = 0, \quad \frac{\partial u}{\partial x}(L,y) = 0$$

$$u(x,0) = 0 \qquad u(x,H) = g(x)$$

Notice that we have 2 homogeneous conditions in the x variable so let $u(x,y) = \phi(x)\, h(y)$ so that

$$h(y)\frac{d^2\phi}{dx^2} + \phi(x)\frac{d^2 h}{dy^2} = 0,$$ then dividing by $h(y)\,\phi(x)$ yeilds

$$\frac{1}{\phi}\frac{d^2\phi}{dx^2} = -\frac{1}{h}\frac{d^2 h}{dy^2} = -\lambda$$ where λ is the separation constant.

We have reduced the problem into 2 odes:

$$\boxed{\begin{aligned} &\frac{d^2\phi}{dx^2} = -\lambda\phi \\ &\frac{d\phi}{dx}(0) = 0, \quad \frac{d\phi}{dx}(L) = 0 \end{aligned}} \quad (P_1) \text{ and } \quad \boxed{\begin{aligned} &\frac{d^2 h}{dy^2} = \lambda h \\ &h(0) = 0 \\ &h(H) = g(x) \end{aligned}} \quad (P_2)$$

From our table of BVPs, (P_1)

~~$\lambda \geq 0$~~

$$\lambda = \left(\frac{n\pi}{L}\right)^2, \quad n = 0, 1, 2, 3 \ldots$$

with eigenfunctions

$$\phi_n(x) = \cos\left(\frac{n\pi}{L}x\right)$$

For P_2,

$\lambda = 0 \Rightarrow h(y) = c_1 y + c_2$

$h(0) = 0 \Rightarrow c_2 = 0$ so

$h(y) = c_1 y$.

$\lambda > 0 \Rightarrow$

$$h(y) = C_n \cosh\left(\frac{n\pi}{L}y\right) + D_n \sinh\left(\frac{n\pi}{L}y\right)$$

$h(0) = 0 \Rightarrow C_n = 0$ because $\sinh(0) = 0$.

so
$$h(y) = D_n \sinh\left(\frac{n\pi}{L}y\right)$$

So for $n = 0$, we have a product solution

$$A_0\,\phi_0(x)\, y$$

$n \geq 1$, we have $A_n\,\phi_n(x)\sinh\left(\frac{n\pi}{L}y\right)$.

By the principle of super-position

$$u(x,y) = A_0 y + \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi}{L} y\right) \cos\left(\frac{n\pi}{L} x\right)$$

To determine the coefficients we use the non-homogeneous b.c $u(x,H) = g(x)$.

Indeed,

$$\boxed{g(x) = u(x,H) = A_0 H + \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi}{L} H\right) \cos\left(\frac{n\pi}{L} x\right)} \quad *$$

multiplying by $(\phi_0(x) = 1)$ to obtain A_0, and integrating yeilds

$$\int_0^L g(x)\,dx = \int_0^L A_0 H\,dx + \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi}{L} H\right) \int_0^L \cos\left(\frac{n\pi}{L} x\right) dx.$$

$$\int_0^L g(x)\,dx = A_0 H L + 0$$

so $$A_0 = \frac{1}{HL} \int_0^L g(x)\,dx.$$

Then multiply by $\phi_m(x) = \cos\left(\frac{m\pi}{L} x\right)$, so that

$$\int_0^L g(x) \cos\left(\frac{m\pi}{L} x\right) dx = A_0 H \int_0^L \cos\left(\frac{m\pi x}{L}\right) dx + \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi}{L} H\right) \int_0^L \cos\left(\frac{n\pi}{L} x\right) \cos\left(\frac{m\pi x}{L}\right) dx$$

so $$A_n = \frac{2}{L \sinh\left(\frac{n\pi}{L} H\right)} \int_0^L \cos\left(\frac{n\pi x}{L}\right) dx.$$

<u>Side node</u>

by comparing coefficients from the Table we know that

if $g(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)$ then $A_0 = \frac{1}{L} \int_0^L g(x)\,dx$, $A_n = \frac{2}{L} \int_0^L g(x) \cos\left(\frac{n\pi x}{L}\right) dx$

so $A_0 H = \frac{1}{L} \int_0^L g(x)\,dx$ and $A_n \sinh\left(\frac{n\pi}{L} H\right) = \frac{2}{L} \int_0^L g(x) \cos\left(\frac{n\pi x}{L}\right) dx.$

5. We want to write

$$|\sin(x)| = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \quad \text{on } [-\pi,\pi].$$

Integrating on both sides

$$\int_{-\pi}^{\pi} |\sin(x)|\, dx = \int_{-\pi}^{\pi} A_0\, dx + \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) dx$$

$\swarrow$

$$\int_{-\pi}^{0} -\sin(x)\, dx + \int_{0}^{\pi} \sin(x)\, dx = A_0 \cdot 2\pi + \sum_{n=1}^{\infty} A_n \int_{-\pi}^{\pi} \cos\left(\frac{n\pi x}{L}\right) dx$$

$$\cos(x)\Big|_{-\pi}^{0} + (-\cos(x))\Big|_{0}^{\pi} = A_0 \cdot 2\pi$$

$$[(1-(-1))] + [(-1)+1] = A_0 \cdot 2\pi \Rightarrow A_0 = \frac{2}{\pi}.$$

$$A_n = -\frac{2}{\pi}\frac{(-1)^n + 1}{n^2 - 1} \quad n \neq 1 \quad , \text{if } n=1 \quad A_1 = 0.$$

$$\frac{2}{\pi} + \sum_{n=2}^{\infty} \left(-\frac{2}{\pi}\frac{(-1)^n + 1}{n^2 - 1}\right)\cos(nx) \quad \text{so } L = \pi$$

$n = 2k.$

If n is odd the coefficient is zero

$$\frac{2}{\pi} + \sum_{k=1}^{\infty} -\frac{2}{\pi}\frac{(-1)^{2k} + 1}{(2k)^2 - 1}\cos(2kx) = |\sin(x)|$$

$$\frac{2}{\pi} + \sum_{k=1}^{\infty} -\frac{2}{\pi}\frac{2}{4k^2 - 1}\cos(2kx) = |\sin(x)|$$

$$\frac{2}{\pi} + \sum_{k=1}^{\infty} -\frac{2}{\pi}\frac{2}{4k^2 - 1} = 0 \qquad \boxed{\sum_{k=1}^{\infty} \frac{1}{4k^2 - 1} = \frac{1}{2}}$$