1. Suppose $f(x)$ and $f'(x)$ are piecewise smooth. Prove that the Fourier cosine series of a continuous $f(x)$ can be differentiated term by term.

PROOF

Suppose $f(x) \sim \sum_{n=0}^{\infty} \beta_n \cos\left(\frac{n\pi x}{L}\right)$, $\beta_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$

Since $f'(x)$ is piecewise smooth,

$$f'(x) \sim A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right).$$

Computing the coefficients :

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f'(x)\,dx = \frac{1}{2L} f(x)\Big|_{-L}^{L} = \frac{1}{2L}\Big(f(L) - f(-L)\Big) = 0$$

because the Fourier cosine series of $f(x)$ is continuous, which implies

$$f(L) = f(-L).$$

$$A_n = \frac{1}{L}\int_{-L}^{L} f'(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\left[f(x)\cos\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{L} + \frac{n\pi}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx\right]$$

$$= \frac{1}{L}\cdot\frac{n\pi}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = 0$$

because $f(x)$ is an even function.

$$B_n = \frac{1}{L}\int_{-L}^{L} f'(x)\sin\left(\frac{n\pi x}{L}\right)dx \underset{\uparrow}{=} -\frac{n\pi}{L}\cdot \underbrace{\frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx}_{=\beta_n}$$

integrating by parts

This is a coefficient in the Fourier cosine series of $f(x)$!

Thus

$$f'(x) \sim -\sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right)\beta_n \sin\left(\frac{n\pi x}{L}\right) = \sum_{n=1}^{\infty} \beta_n \frac{d}{dx}\left(\cos\left(\frac{n\pi x}{L}\right)\right)$$

hence the Fourier cosine series for $f(x)$ can be differentiated term by term to get the Fourier sine series for $f'(x)$.

3. $$\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2} + \sin(3\pi x)$$

$$u(0,t) = 0 \quad , \quad u(1,t) = 0$$

$$u(x,0) = \sin(\pi x)$$

Assuming u, u_t, u_x, u_{xx} are piecewise smooth and continuous.

Expand $u(x,t)$ in terms of the eigen functions so

$$u(x,t) \sim \sum_{n=1}^{\infty} B_n(t)\sin\left(\frac{n\pi x}{L}\right) \qquad (L=1)$$

The initial condition is satisfied if

$$f(x) = \sin(\pi x) \sim \sum_{n=1}^{\infty} B_n(0)\sin\left(\frac{n\pi x}{L}\right) \qquad (L=1)$$

Comparing coefficients yelds $B_n(0) = \begin{cases} 1, & n=1 \\ 0, & \text{otherwise.} \end{cases}$

If $u(x,t)$ is continuous, the Fourier sine series can be differentiated term by term provided $u(0,t)=0$ and $u(1,t)=0$ (the boundary conditions)

$$\frac{\partial u}{\partial x} \sim -\sum_{n=1}^{\infty}\left(\frac{n\pi}{4}\right) B_n(t)\cos\left(n\pi x\right)$$

Since $\frac{\partial u}{\partial x}$ is also continuous, the Fourier cosine series can be differentiated term by term so

$$\frac{\partial^2 u}{\partial x^2} \sim -\sum_{n=1}^{\infty}(n\pi)^2 B_n(t)\sin(n\pi x)$$

u_t is piecewise smooth so

$$\frac{\partial u}{\partial t} \sim \sum_{n=1}^{\infty}\frac{d}{dt}B_n(t)\sin(n\pi x) \quad , \text{ plug into PDE}$$

$$\sum_{n=1}^{\infty}\frac{d}{dt}B_n(t)\sin(n\pi x) = -4\sum_{n=1}^{\infty}(n\pi)^2 B_n(t)\sin(n\pi x) + \sin(3\pi x) \quad (*)$$

Comparing coefficients in (*), we have

$n=3$

$$B_n'(t) = -4(n\pi)^2 B_n(t) + 1$$

$n \neq 3$

$$B_n'(t) = -4(n\pi)^2 B_n(t)$$

Solving

$n=3$

~~$B_3'(t) + 4(n\pi)^2 B_n(t) = 1$, multiply by integrating factor~~ e

$$B_3'(t) = \cancel{-4(3\pi)^2} - 36\pi^2 B_n(t) + 1 \Rightarrow B_3'(t) + 36\pi^2 B_3(t) = 1$$

Recall that

$$y(t)e^{kt} = \int_0^t q(w)e^{kw} + y(0)$$

$$y' + ky = q(t).$$

Integrating factor

$$B_3(t)e^{36\pi^2 t} = \int_0^t e^{36\pi^2 w}\,dw + B_3(0)$$

$$= \frac{1}{36\pi^2} e^{36\pi^2 w}\Big|_0^t + B_3(0)$$

$$= \frac{1}{36\pi^2}\left[e^{36\pi^2 t} - 1\right] + B_3(0)$$

$$B_3(t) = e^{-36\pi^2 t}\left[\frac{1}{36\pi^2}\left(e^{36\pi^2 t} - 1\right) + B_3(0)\right]$$

$n \neq 3$

$$B_n(t) = B_n(0)e^{-4(n\pi)^2 t}$$

$$e^x \sim A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)$$

because we have equality on $(0,L)$.

$$\frac{d}{dx}(e^x) = e^x \sim -\sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right)\sin\left(\frac{n\pi x}{L}\right).$$

To differentiate once more, use (3.4.13)

$$\frac{d^2}{dx^2}(e^x) = e^x \sim \frac{d}{dx}\left[-\sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right)A_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

$$= \frac{1}{L}\left(e^L - e^0\right) + \sum_{n=1}^{\infty}\left[\left(\frac{n\pi}{L}\right)\left(-\frac{n\pi}{L}A_n\right) + \frac{2}{L}\left((-1)^n e^L - e^0\right)\right]\cos\left(\frac{n\pi x}{L}\right)$$

$$= \frac{1}{L}\left(e^L - e^0\right) + \sum_{n=1}^{\infty}\left[\frac{-n^2\pi^2}{L^2}A_n + \frac{2}{L}\left((-1)^n e^L - 1\right)\right]\cos\left(\frac{n\pi x}{L}\right)$$

$$= A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)$$

Comparing coefficients

$$A_0 = \frac{1}{L}\left(e^L - e^0\right) \quad \text{and} \quad A_n = \frac{-n^2\pi^2}{L^2}A_n + \frac{2}{L}\left((-1)^n e^L - 1\right)$$

$$\left(1 + \frac{n^2\pi^2}{L^2}\right)A_n = \frac{2}{L}\left((-1)^n e^L - 1\right)$$

$$A_n = \frac{\frac{2}{L}\left((-1)^n e^L - 1\right)}{1 + \frac{n^2\pi^2}{L^2}} = \frac{2}{L}\cdot\frac{L^2}{L^2+n^2\pi^2}\left((-1)^n e^L - 1\right)$$

$$= \frac{2L\left((-1)^n e^L - 1\right)}{L^2 + n^2\pi^2}$$