Suppose \( f(x) \) and \( f'(x) \) are piecewise smooth. Prove that the Fourier cosine series of a continuous \( f(x) \) can be differentiated term by term.

**Proof:**

Suppose \( f(x) \sim \sum_{n=0}^{\infty} b_n \cos \left( \frac{n\pi x}{L} \right) \), \( b_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \left( \frac{n\pi x}{L} \right) dx \).

Since \( f'(x) \) is piecewise smooth,

\[
f'(x) \sim A_0 + \sum_{n=1}^{\infty} A_n \cos \left( \frac{n\pi x}{L} \right) + \sum_{n=1}^{\infty} B_n \sin \left( \frac{n\pi x}{L} \right).
\]

Computing the coefficients:

\[
A_0 = \frac{1}{2L} \int_{-L}^{L} f(x) \, dx = \frac{1}{2L} f(x) \bigg|_{-L}^{L} = \frac{1}{2L} \left( f(L) - f(-L) \right) = 0
\]

because the Fourier cosine series of \( f(x) \) is continuous, which implies \( f(L) = f(-L) \).

\[
A_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \left( \frac{n\pi x}{L} \right) \, dx = \frac{1}{L} \left[ f(x) \cos \left( \frac{n\pi x}{L} \right) \right]_{-L}^{L} + \frac{n\pi}{L} \int_{-L}^{L} f(x) \sin \left( \frac{n\pi x}{L} \right) \, dx
\]

\[= \frac{1}{L} \cdot \frac{n\pi}{L} \int_{-L}^{L} f(x) \sin \left( \frac{n\pi x}{L} \right) \, dx = 0
\]

because \( f(x) \) is an even function.

\[
B_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \left( \frac{n\pi x}{L} \right) \, dx = -\frac{n\pi}{L} \cdot \frac{1}{L} \int_{-L}^{L} f(x) \cos \left( \frac{n\pi x}{L} \right) \, dx
\]

\[
\text{Integrating by parts} \quad \text{This is a coefficient in the Fourier cosine series of } f(x).
\]

Thus

\[
f'(x) \sim -\sum_{n=1}^{\infty} \left( \frac{n\pi}{L} \right) b_n \sin \left( \frac{n\pi x}{L} \right) = \sum_{n=1}^{\infty} \frac{\beta_n}{L} \, d \left( \cos \left( \frac{n\pi x}{L} \right) \right)
\]

hence the Fourier cosine series for \( f'(x) \) can be differentiated term by term to get the Fourier sine series for \( f'(x) \).
3. \[
\frac{du}{dt} = 4 \frac{d^2u}{dx^2} + \sin(3\pi x)
\]

\[
u(0,b) = 0 \quad , \quad u(1,b) = 0
\]

\[
u(x,0) = \sin(\pi x)
\]

Assuming \(\nu, \nu_t, \nu_x, \nu_xx\) are piecewise smooth and continuous.

Expand \(\nu(x,t)\) in terms of the eigenfunctions so

\[
\nu(x,t) \sim \sum_{n=1}^{\infty} B_n(b) \sin\left(\frac{n\pi x}{L}\right) \quad (L=1)
\]

The initial condition is satisfied if

\[
f(x) = \sin(\pi x) \sim \sum_{n=1}^{\infty} B_n(x) \sin\left(\frac{n\pi x}{L}\right) \quad (L=1)
\]

Comparing coefficients yields \(B_n(x) = \begin{cases} 1, & n=1 \\ 0, & \text{otherwise} \end{cases}\)

If \(\nu(x,t)\) is continuous, the Fourier sine series can be differentiated term by term provided \(\nu(0,t) = 0\) and \(\nu(1,t) = 0\) (the boundary conditions)

\[
\frac{du}{dx} \sim -\sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right) B_n(b) \cos\left(\frac{n\pi x}{L}\right)
\]

Since \(\frac{du}{dx}\) is also continuous, the Fourier cosine series can be differentiated term by term in

\[
\frac{d^2u}{dx^2} \sim -\sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right)^2 B_n(b) \sin\left(\frac{n\pi x}{L}\right)
\]

\(\nu_t\) is piecewise smooth so

\[
\frac{du}{dt} \sim \sum_{n=1}^{\infty} \frac{d}{dt} B_n(b) \sin\left(\frac{n\pi x}{L}\right), \quad \text{plug into PDE}
\]

\[
\sum_{n=1}^{\infty} \frac{d}{dt} B_n(b) \sin\left(\frac{n\pi x}{L}\right) = -4 \sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right)^2 B_n(b) \sin\left(\frac{n\pi x}{L}\right) + \sin(3\pi x) \quad (\star)
\]
Comparing coefficients in $u(x)$, we have

\[ n = 3 \]
\[ B_n(t) = -4(n\pi)^2 B_{n-1}(t) + 1 \]

\[ n \neq 3 \]
\[ B_n'(t) = -4(n\pi)^2 B_n(t) \]

Solving for $n = 3$

\[ B_3'(t) = -4(3\pi)^2 - 36\pi^2 B_3(t) + 1 \Rightarrow B_3'(t) + 36\pi^2 B_3(t) = 1 \]

Integrating factor

\[ B_3(t) e^{36\pi^2 t} = \int_0^t e^{36\pi^2 \omega} d\omega + B_3(0) \]

\[ = \frac{1}{36\pi^2} e^{36\pi^2 t} \bigg|_0^t + B_3(0) \]

\[ = \frac{1}{36\pi^2} \left[ e^{36\pi^2 t} - 1 \right] + B_3(0) \]

\[ B_3(t) = e^{-36\pi^2 t} \left[ \frac{1}{36\pi^2} \left( e^{36\pi^2 t} - 1 \right) + B_3(0) \right] \]

\[ n \neq 3 \]
\[ B_n(t) = B_{n}(t) e^{-4(n\pi)^2 t} \]
\[ e^x \sim A_0 + \sum_{n=1}^{\alpha} A_n \cos \left( \frac{n\pi x}{L} \right) \]

because we have equality on \((0, L)\),

\[ \frac{d}{dx} (e^x) = e^x \sim - \sum_{n=1}^{\alpha} \left( \frac{n\pi}{L} \right) \sin \left( \frac{n\pi x}{L} \right). \]

To differentiate once more, use (3.4.13)

\[ \frac{d^2}{dx^2} (e^x) = e^x \sim \frac{d}{dx} \left[ - \sum_{n=1}^{\alpha} \left( \frac{n\pi}{L} \right) A_n \sin \left( \frac{n\pi x}{L} \right) \right] = \frac{1}{L} (e^L - e^0) + \sum_{n=1}^{\alpha} \left[ \left( \frac{n\pi}{L} \right) \left( -\frac{n\pi}{L} A_n \right) + \frac{2}{L} \left( -1 \right)^n e^0 \right] \cos \left( \frac{n\pi x}{L} \right) \]

\[ = \frac{1}{L} (e^L - e^0) + \sum_{n=1}^{\alpha} \left[ -\frac{n^2 \pi^2}{L^2} A_n + \frac{2}{L} \left( -1 \right)^n \right] \cos \left( \frac{n\pi x}{L} \right) \]

\[ = A_0 + \sum_{n=1}^{\alpha} A_n \cos \left( \frac{n\pi x}{L} \right) \]

Comparing coefficients

\[ A_0 = \frac{1}{L} (e^L - e^0) \quad \text{and} \quad A_n = \frac{-n^2 \pi^2}{L^2} A_n + \frac{2}{L} \left( -1 \right)^n e^L \]

\[ \left( 1 + \frac{n^2 \pi^2}{L^2} \right) A_n = \frac{2}{L} \left( -1 \right)^n e^L \]

\[ A_n = \frac{\frac{2}{L} \left( -1 \right)^n e^L}{1 + \frac{n^2 \pi^2}{L^2}} \quad \frac{2 L^2 \left( -1 \right)^n e^{L-1}}{L^2 + n^2 \pi^2} \]

\[ = \frac{2 L \left( -1 \right)^n e^{L-1}}{L^2 + n^2 \pi^2} \]