Practice problems (partial solutions)

(i)

2.5.1 c

Separation of variables - let $u(x,y) = h(x)\,\phi(y)$. Your eigen functions are

$\phi(y) = \sin\left(\frac{n\pi y}{H}\right)$ $\quad n = 1, 2, 3, \ldots$

The x-dependent problem should have solution $h(x) = \cosh\left(\frac{n\pi x}{H}\right)$ after applying $\frac{dh}{dx}(0) = 0$.

By the principle of super-position,

$$u(x,y) = \sum_{n=1}^{\infty} A_n \cosh\left(\frac{n\pi x}{H}\right) \sin\left(\frac{n\pi y}{H}\right)$$

The non-homogeneous condition @ $x = L$ yeilds

$$g(y) = \sum_{n=1}^{\infty} A_n \cosh\left(\frac{n\pi L}{H}\right) \sin\left(\frac{n\pi y}{H}\right) \quad \text{so}$$

we can use that to determine coefficients. Indeed,

$$A_n \cosh\left(\frac{n\pi L}{H}\right) = \frac{2}{L} \int_0^H g(y) \sin\left(\frac{n\pi y}{H}\right) dy.$$

2.5.1 e

Let $u(x,y) = \phi(x)\,h(y) \Rightarrow \lambda = \left(\frac{n\pi}{L}\right)^2$ with $\phi_n(x) = \sin\left(\frac{n\pi x}{L}\right)$

$n = 1, 2, 3, \ldots$

The y-dependent problem has a general solution

$$h = c_1 \cosh\left(\frac{n\pi y}{L}\right) + c_2 \sinh\left(\frac{n\pi y}{L}\right)$$

apply the boundary condition $h(0) - \frac{dh}{dy}(0) = 0$ to get $c_1 = c_2 \frac{n\pi}{L}$.

then $h_n(y) = c_1 \cosh\left(\frac{n\pi y}{L}\right) + c_1 \frac{L}{n\pi} \sinh\left(\frac{n\pi y}{L}\right)$

Superposition yeilds

$$u(x,y) = \sum_{n=1}^{\infty} A_n h_n(y) \sin\left(\frac{n\pi x}{L}\right)$$

where $A_n h_n(H) = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx.$

(2) 2.5.5a

The eigenvalue problem is $\frac{d^2\phi}{d\theta^2} = -\lambda\phi$

$$\frac{d\phi}{d\theta}(0) = 0$$

$$\phi\left(\frac{\pi}{2}\right) = 0$$

It can be shown that $\lambda > 0$ so that $\phi = \cos\sqrt{\lambda}\theta$ where $\phi\left(\frac{\pi}{2}\right) = 0$

$\Rightarrow \cos\left(\sqrt{\lambda}\frac{\pi}{2}\right) = 0$ or $\sqrt{\lambda}\frac{\pi}{2} = -\frac{\pi}{2} + n\pi$, $n = 1, 2, 3, \ldots$

so $\lambda = (2n-1)^2$

The radially dependent problem satisfies

$\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = \lambda$ and hence by the boundedness of the solution at zero

$$G(r) = r^{2n-1}$$

Superposition yeilds

$$u(r,\theta) = \sum_{n=1}^{\infty} A_n r^{2n-1} \cos(2n-1)\theta$$

Apply non-homogeneous condition $f(\theta) = \sum_{n=1}^{\infty} A_n \cos(2n-1)\theta$ to obtain

$$A_n = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} f(\theta)\cos(2n-1)\theta \, d\theta.$$

2.5.5c)

The boundary conditions of (2.5.37) must be replaced by $\phi(0)=0$ & $\phi(\frac{\pi}{2})=0$

Thus $L=\frac{\pi}{2}$ so $\lambda = \left(\frac{n\pi}{L}\right)^2 = (2n)^2$, $\phi = \sin\left(\frac{n\pi\theta}{L}\right) = \sin(2n\theta)$.

The radial part that remains bounded @ $r=0$ is $G = r^{\sqrt{\lambda}} = r^{2n}$

by superposition $$u(r,\theta) = \sum_{n=1}^{\infty} A_n r^{2n} \sin(2n\theta)$$

Use the non-homogeneous condition to compute your coefficients!

(3) 3.3.18

Assume $f(x)$ is continuous,

(a) $f(x)$ equals its Fourier sine series for all $-L \leq x \leq L$ if $f(x)$ is piecewise smooth and $f(-L)=f(L)$

(b) $f(x)$ equals its Fourier sine series for all $0 \leq x \leq L$ if $f(x)$ is piecewise smooth and $f(0)=f(L)=0$

(c) $f(x)$ equals its Fourier cosine series for all $0 \leq x \leq L$ if $f(x)$ is piecewise smooth.

Problems 4-7 See Text or your notes for proofs.

8. 3.4.12 We solved this problem in class - See notes.

3.5.1 (a)

From (3.3.11), (3.3.12)

$$x = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \qquad -L < x < L. \qquad (*)$$

where $B_n = \frac{2L}{n\pi}(-1)^{n+1}$

From (3.5.6),

$$\frac{x^2}{2} = \frac{L}{2}x - \frac{4L^2}{\pi^3}\left(\sin\left(\frac{\pi x}{L}\right) + \frac{\sin\left(\frac{3\pi x}{L}\right)}{3^3} + \frac{\sin\left(\frac{5\pi x}{L}\right)}{5^3} + \dots\right) \quad (**)$$

now take $(*)$ and plug it into $(**)$ ~~to~~ and do some algebra

to write $x^2 \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \quad (***)$

$(***)$ will hold true for $0 < x < L$ because the periodic extension will have a jump discontinuity

(c) See text for solution.

3.5.6

From equation (3.5.6) in text we have

$$\frac{x^2}{2} = \frac{L}{2}x - \frac{4L}{\pi^3}\left(\sin\left(\frac{\pi x}{L}\right) + \frac{\sin\left(\frac{3\pi x}{L}\right)}{3^3} + \frac{\sin\left(\frac{5\pi x}{L}\right)}{5^3} + \dots\right)$$

Let $x = \frac{L}{2}$, evaluate.

The sum of the series is $\frac{\pi^3}{32}$.