Sturm - Liouville Eigenvalue Problems

So far

Method of separation of variables $\longrightarrow$ determines eigenvalues $\longrightarrow$ $\dfrac{d^2\phi}{dx^2} = -\lambda \phi$ $\longrightarrow$ Explicit solutions → eigenfunctions + superposition $\Rightarrow$ Infinite Series

GOAL

Generalize our results to cases where we may not solve the ODE.

Examples.

Heat flow in a non-uniform rod

$$c\rho \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0 \frac{\partial u}{\partial x}\right) + Q \qquad (*).$$

* Thermal coefficients are allowed to vary in x.

Let $Q = \alpha u$, so that

$$c\rho \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0 \frac{\partial u}{\partial x}\right) + \alpha u ,$$ where $\alpha(x)$

Even though $Q \neq 0$, the choice $Q = \alpha u$ means that the PDE is still linear and homogeneous.

Physical situations in which $Q = \alpha u$ arises

① A chemical reaction that generates heat (reaction is more intense at higher temperature) $Q > 0$ exothermic

If we have homogeneous boundary conditions @ $x=0$ and $x=L$

Separation of variables

Let $u(x,t) = \phi(x)\,h(t)$

$$c\rho\,\phi(x)\,\frac{dh}{dt} = h(t)\,\frac{d}{dx}\left(K_0\,\frac{d\phi}{dx}\right) + \alpha\,\phi(x)\,h(t)$$

divide by $c\rho\,\phi(x)\,h(t)$

$$\frac{1}{h}\,\frac{dh}{dt} = \frac{1}{c\rho\,\phi}\,\frac{d}{dx}\left(K_0\,\frac{d\phi}{dx}\right) + \frac{\alpha}{c\rho} = -\lambda$$

$\lambda > 0$ (exponentially decaying solutions)

$\lambda < 0$ (exponentially growing solutions) $\quad \lambda = 0$ constant.

If $\alpha > 0$ ($\lambda < 0$ is possible if energy is being added to the rod).

Spatial ODE

$$\frac{d}{dx}\left(K_0\,\frac{d\phi}{dx}\right) + \alpha\phi + \lambda c\rho\phi = 0. \qquad (*)$$

K_0, ρ, c, α are all space dependent so we cannot in general solve $(*)$ by hand. (We can only analyze this problem qualitatively)

Sturm-Liouville Eigenvalue problems

General classification (Sturm-Liouville ODE) mid 1800s.

$$\frac{d}{dx}\left(p\,\frac{d\phi}{dx}\right) + q\phi + \lambda\sigma\phi = 0, \qquad a < x < b.$$

where λ is the eigenvalue.

Boundary conditions

$\phi = 0$ Dirichlet boundary condition.

Heat flow	Vibrating String
Fixed (zero) temp	Fixed (zero) displacement

$\dfrac{d\phi}{dx} = 0$ Neumann Condition — Insulated — Free

Periodic

$\phi(-L) = \phi(L)$

$\dfrac{d\phi}{dx}(-L) = \dfrac{d\phi}{dx}(L)$

Perfect thermal contact —

General form

$$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi + \lambda\sigma(x)\phi = 0, \quad a < x < b$$

Subject to

$$\beta_1 \phi(a) + \beta_2 \frac{d\phi}{dx}(a) = 0$$

$$\beta_3 \phi(b) + \beta_4 \frac{d\phi}{dx}(b) = 0.$$

where
- β_i are real.
- p, q and σ are real and continuous everywhere.
- $p > 0$, $\sigma > 0$.

} Regular Sturm-Liouville eigenvalue problem

Theorem

For the general Sturm-Liouville problem,

1. All eigenvalues are real
2. There exist an infinite number of eigenvalues

$$\lambda_1 < \lambda_2 < \dots < \lambda_n < \lambda_{n+1} < \dots$$

* λ_1 is the smallest

$\lambda_n \to \infty$ as $n \to \infty$.

3. Corresponding to each λ_n, there is an eigenfunction $\phi_n(x)$ (unique upto an additive constant) $\phi_n(x)$ has $n-1$ zeros on $a < x < b$.

4. $\phi_n(x)$ form a complete set, i.e any piecewise smooth function can be represented as a Fourier series of eigenfunctions

$$f(x) \sim \sum_{n=1}^{\infty} a_n \phi_n(x)$$

* converges to $\frac{1}{2}\left[f(x_+) + f(x_-)\right]$.

5. Eigenfunctions are orthogonal relative to $\sigma(x)$

$$\int_a^b \phi_n(x)\, \phi_m(x)\, \sigma(x)\, dx = 0 \quad \text{if } \lambda_n \neq \lambda_m,$$

6. Any eigenvalue can be related to its eigenfunction by

$$\boxed{\lambda = \frac{-p\phi \frac{d\phi}{dx}\Big|_a^b + \int_a^b \left(p\left(\frac{d\phi}{dx}\right)^2 - q\phi^2\right)dx}{\int_a^b \phi^2 \sigma\, dx}}$$

Rayleigh Quotient.

<u>Illustrations of Theorems</u>

1. Simplest regular Sturm-Liouville problem

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0 \qquad (P)$$

$$\phi(0) = \phi(L) = 0$$

We already know that $\lambda_n = \left(\frac{n\pi}{L}\right)^2$, $\phi_n(x) = \sin\left(\frac{n\pi x}{L}\right)$ $n = 1, 2, 3, \ldots$

From the theorem, the eigenvalues of (P) have to be real. We did not have to check $\lambda \in \mathbb{C}$. We checked $\lambda < 0$, $\lambda > 0$, $\lambda = 0$.

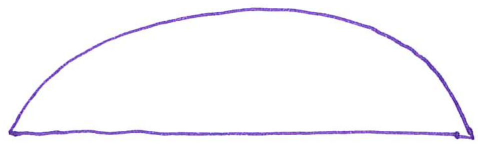
2. Ordering of eigenvalues

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad \lambda_n \to \infty \text{ as } n \to \infty$$

This is valid for any regular Sturm-Liouville problem

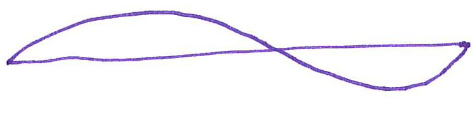
3. Zeros of eigenfunctions

$$\phi_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$



$n=1$

0 L



$n=2$

⋮

4. The eigenfunctions can be used to represent any piecewise smooth, $f(x)$

$$f(x) \sim \sum_{n=1}^{\infty} a_n \phi_n(x)$$

For $\phi_n(x) = \sin\left(\frac{n\pi x}{L}\right)$, this is a Fourier sine series!

5. <u>Orthogonality of eigenfunctions</u>

The eigenfunctions of any regular Sturm-Liouville eigenvalue problem will always be orthogonal.

$$\int_a^b \phi_n(x)\, \phi_m(x)\, \sigma(x)\, dx = 0 \qquad \text{if } \lambda_m \neq \lambda_n.$$

In the case of (P) $\sigma(x) = 1$

In the general case

$$f(x)\, \phi_m(x)\, \sigma(x) = \sum_{n=1}^{\infty} a_n \phi_n(x)\, \phi_m(x)\, \sigma(x)$$

$$\int_a^b f(x)\, \phi_m(x)\, \sigma(x) = \sum_{n=1}^{\infty} a_n \int \phi_n(x)\, \phi_m(x)\, \sigma(x)\, dx$$

so that

$$a_m = \frac{\int_a^b f(x)\,\phi_m(x)\,\sigma(x)}{\int_a^b \phi_m^2\,\sigma(x)\,dx}$$

6. <u>Rayleigh Quotient</u>.

$$\frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right] + q(x)\,\phi + \lambda\,\sigma(x)\,\phi = 0 \quad , \quad a<x<b.$$

multiply by ϕ and integrate

$$\int_a^b \left[\phi\,\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q\phi^2\right]dx + \lambda\int_a^b \phi^2\sigma\,dx = 0$$

solving for λ $\left(\int_a^b \phi^2\sigma\,dx > 0\right)$

$$\lambda = -\frac{\int_a^b\left[\phi\,\frac{d}{dx}\left(p\frac{d\phi}{dx}\right) + q\phi^2\right]dx}{\int_a^b \phi^2\sigma\,dx}$$

Integrating by parts

$$dv = \frac{d}{dx}\left(p\frac{d\phi}{dx}\right)dx \qquad v = p\frac{d\phi}{dx}$$

$$u = \phi \qquad\qquad du = \cancel{dx}\,\frac{d\phi}{dx}$$

$$\int_a^b\left[\phi\,\frac{d}{dx}\left(p\frac{d\phi}{dx}\right)\right]\cancel{q}\,dx = \phi p\frac{d\phi}{dx}\Big|_a^b - \int_a^b p\left(\frac{d\phi}{dx}\right)^2$$

$$\lambda = \frac{-p\phi\frac{d\phi}{dx}\Big|_a^b + \int_a^b\left[p\left(\frac{d\phi}{dx}\right)^2 - q\phi^2\right]dx}{\int_a^b \phi^2\sigma\,dx} \qquad (*)$$

(*) is is never directly used because you have to know the eigenvalues to evaluate the eigenfunctions.

<u>However!</u>

It is a very useful mathematical tool.

e.g let $a=0$, $b=L$, $p(x)=1$, $q(x)=0$, $\sigma(x)=1$ and $\phi(0)=0$, $\phi(L)=0$ then

$$\lambda = \frac{\int_0^L \left(\frac{d\phi}{dx}\right)^2 dx}{\int_0^L \phi^2 dx} \qquad (**)$$

Since both the numerator and denominator ≥ 0 $(\phi > 0)$ the denominator cannot be zero. It follows that $\lambda \geq 0$ without solving the ODE.

<u>$\lambda = 0$ case</u> $\left(\phi(0) = \phi(L) = 0\right)$

from $(**)$ $\lambda = 0$ if and only if $\frac{d\phi}{dx} \equiv 0$ $\forall x$ so

ϕ must be a constant function satisfying $\phi(0) = \phi(L) = 0$

so $\phi(x) \equiv 0$ and thus $\phi(x)$ cannot be an eigenfunction!

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, \ldots$$

Heat flow in a non-uniform rod

$$c\rho \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0 \frac{\partial u}{\partial x}\right)$$

$$u(0,t) = 0$$

$$\frac{\partial u}{\partial x}(L,t) = 0$$

$$u(x,0) = f(x)$$

From separation of variables, $u(x,t) = \phi(x)\,h(t)$

① $\frac{dh}{dt} = -\lambda h$ (time dependent problem)

② $\frac{d}{dx}\left(K_0 \frac{d\phi}{dx}\right) + \lambda c\rho\phi = 0$

$$\phi(0) = 0$$

$$\frac{d\phi}{dx}(L) = 0$$

① $h(t) = c e^{-\lambda_n t}$

② From our Sturm-Liouville theory, there is an infinite number of eigenvalues λ_n corresponding to eigenfunctions $\phi_n(x)$

so

$$u(x,t) = \phi_n(x)\, e^{-\lambda_n t}$$

by the principle of superposition,

$$u(x,t) = \sum_{n=1}^{\infty} a_n \phi_n(x)\, e^{-\lambda_n t}$$

From the initial conditions,

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} a_n \phi_n(x) \qquad (*)$$

From our theory, we can represent any piecewise smooth function as $(*)$

The eigenfunctions are orthogonal with a weight $\sigma(x) = c(x)\,\rho(x)$.

$$\int_0^L f(x)\,\phi_m(x)\,c(x)\,\rho(x)\,dx = \sum_{n=1}^{\infty} a_n \int_0^L \phi_n(x)\,\phi_m(x)\,c(x)\,\rho(x)\,dx$$

so $$a_n = \frac{\int_0^L f(x)\,\phi_n(x)\,c(x)\,\rho(x)\,dx}{\int_0^L \phi_n^2(x)\,c(x)\,\rho(x)\,dx}.$$

Qualitative analysis of the solution

~~1. $\{\phi_n\}$ is an increasing sequence so $\int_0^L \phi_n^2(x)\,c(x)\,\rho(x)\,dx \to \infty$ as $n \to \infty$~~

~~so $a_n \to 0$.~~

Our solution $u(x,t) = \sum_{n=1}^{\infty} a_n \phi_n(x)\, e^{-\lambda_n t}$ (heat equation with no sources + Zero dirichlet & Neumann)

should be bounded in time so as $t \to \infty$ $\left(a_n \phi_n e^{-\lambda_n t}\right)$ should get smaller.

so we can approximate the solution by

$$u(x,t) \cong a_1 \phi_1(x)\, e^{-\lambda_1 t}$$

provided $a_1 \neq 0$.

But $$a_1 = \frac{\int_0^L f(x)\,\phi_1(x)\,c(x)\,\rho(x)\,dx}{\int_0^1 \phi_1^2(x)\,c(x)\,\rho(x)\,dx} \neq 0$$ because $\phi_1(x)$ is an eigenfunction (and thus $\phi_1(x) \neq 0$), in addition $c(x)$, $\rho(x)$ are positive physical functions.

Thus, if $f(x) > 0$ it follows that $a_1 \neq 0$

* For large t, the temperature distribution remains relatively the same in time.

** The amplitude grows / decays depending on the sign of λ_1

In this particular case

① No sources

② Right side is insulated but $u=0$ on the left.

so we expect the solution to decay exponentially to zero. $(\lambda > 0)$

Proof

Rayleigh Quotient

$$\boxed{\lambda = \frac{\int_0^L K_0(x)\left(\frac{d\phi}{dx}\right)^2 dx}{\int_0^L \phi^2 c(x)\, \rho(x)\, dx.}} \quad (R)$$

$-K_0 \phi \frac{d\phi}{dx}\Big|_a^L$ vanishes due to boundary conditions.

Recall that in general, the Eigenvalue problem is

$$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi + \lambda\sigma(x)\phi = 0 \qquad a < x < b$$

with

$$\lambda = \frac{-p\phi\frac{d\phi}{dx}\Big|_a^b + \int_a^b \left[p\left(\frac{d\phi}{dx}\right)^2 - q\phi^2\right]dx}{\int_a^b \phi^2 \sigma\, dx}$$

From (R) $\lambda > 0$ because the thermal coefficients are positive and $\phi > 0$ as $\phi = \text{constant}$ is the zero function $\left(\phi(0) = 0 \text{ and } \frac{d\phi}{dx}(L) = 0\right)$.

We conclude that

$$\lim_{t\to\infty} u(x,t) = 0.$$

because $u(x,t) = \sum_{n=1}^{\infty} a_n \phi_n(x)\, e^{-\lambda_n t}$

Motivation (Self-adjoint operators and Sturm-Liouville problems)

Prove some properties of the regular Sturm-Liouville eigenvalue problem.

$$\frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right] + q(x)\phi(x) + \lambda\sigma(x)\phi = 0$$

$$\beta_1\phi(a) + \beta_2\frac{d\phi}{dx}(a) = 0$$

$$\beta_3\phi(b) + \beta_4\frac{d\phi}{dx}(b) = 0.$$

* Recall that regular means p, q and σ are real and continuous everywhere
$p > 0$, $\sigma > 0$.

We will prove

① Orthogonality of eigenfunctions

② Eigenvalues are real

③ Comment on uniqueness of eigenfunctions.

Self - adjoint operators and Sturm - Liouville eigenvalue problems

Let

$$L(y) = \frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)\,y$$

then the SL eigenvalue problem

$$\frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right] + q(x)\phi + \lambda\sigma(x)\phi = 0$$ can be written as

$$L(\phi) + \lambda\sigma(x)\phi = 0.$$ where

λ is the eigenvalue and ϕ is an eigenfunction.

Lagrange Identity.

Consider $uL(v) - vL(u)$ for any 2 functions u and v (not necessarily eigenfunctions)

$uL(v) - v(L(u))$

$$u\left\{\frac{d}{dx}\left(p(x)\frac{dv}{dx}\right) + q(x)v\right\} - v\left\{\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u\right\}$$

$$= \boxed{u\frac{d}{dx}\left(p(x)\frac{dv}{dx}\right)} - \boxed{v\frac{d}{dx}\left(p(x)\frac{du}{dx}\right)}$$

= Notice that

$$\frac{d}{dx}\left(up\frac{dv}{dx}\right) = \boxed{u\frac{d}{dx}\left(p(x)\frac{dv}{dx}\right)} + p\frac{dv}{dx}\cdot\frac{du}{dx} \Rightarrow u\frac{d}{dx}\left(p\frac{dv}{dx}\right) = \frac{d}{dx}\left(up\frac{dv}{dx}\right) - p\frac{dv}{dx}\cdot\frac{du}{dx}$$

and $$\frac{d}{dx}\left(vp\frac{du}{dx}\right) = \boxed{v\frac{d}{dx}\left(p(x)\frac{du}{dx}\right)} + p\frac{du}{dx}\cdot\frac{dv}{dx}$$

so $$\left[\frac{d}{dx}\left(up\frac{dv}{dx}\right) - \underbrace{p\frac{dv}{dx}\cdot\frac{du}{dx}}\right] - \left[\frac{d}{dx}\left(vp\frac{du}{dx}\right) - \underbrace{p\frac{du}{dx}\cdot\frac{dv}{dx}}\right]$$

These terms are the same

$$= \frac{d}{dx}\left(p\left[u\frac{dv}{dx} - v\frac{du}{dx}\right]\right)$$

$$\Rightarrow \quad u\mathcal{L}(v) - v\mathcal{L}(u) = \frac{d}{dx}\left(p\left[u\frac{dv}{dx} - v\frac{du}{dx}\right]\right) \qquad (\mathcal{L})$$

Integrating $(\mathcal{L})$ on both sides.

GREEN'S FORMULA

$$\boxed{\int_a^b [uL(v) - vL(u)]\,dx = p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Bigg|_a^b}$$

Self-adjointness

Suppose u and v have the additional property

$$p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Bigg|_a^b = 0$$

then

$$\int_a^b [uL(v) - vL(u)]\,dx = 0 \qquad (*)$$

Vanishing of boundary term

1° If u and v both satisfy homogeneous boundary conditions

$$\phi(a) = 0$$

$$\frac{d\phi}{dx}(b) + h\phi(b) = 0$$

When $(*)$ is valid, the operator is called self-adjoint.

Side node

Let $T \in \mathcal{L}(V,W)$, V and W are inner product spaces

an adjoint operator $T^* \in \mathcal{L}(W,V)$ is one such that

$$\langle Tv, w\rangle = \langle v, T^*, w\rangle$$

2° Periodic boundary conditions

$$\phi(a) = \phi(b), \quad p(a)\frac{d\phi}{dx}(a) = p(b)\frac{d\phi}{dx}(b).$$

3. Singular case

$p(x) = 0$ at an endpoint.

Using Green's formula to prove orthorgonality of eigenfunctions

* For the (SLP) eigenfunctions are orthorgonal with a weight $\sigma(x)$

i.e $$\int_a^b \phi_n(x)\,\phi_m(x)\,\sigma(x)\,dx = 0 \quad \text{if } \lambda_n \neq \lambda_m.$$

Let λ_n and λ_m be eigenvalues with corresponding eigenfunctions $\phi_n(x)$ and $\phi_m(x)$.

The ODEs satisfied by the eigenfunctions are

$$L(\phi_n) + \lambda_n \sigma(x)\phi_n = 0 \qquad (P)$$
$$L(\phi_m) + \lambda_m \sigma(x)\phi_m = 0$$

where $$L(y) = \frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y.$$

In addition $\phi_n(x)$ and $\phi_m(x)$ satisfy homogeneous boundary conditions.

Recall Green's formula

$$\int_a^b [u L(v) - v L(u)]\,dx = p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_a^b$$

u and v are arbitrary, so let $u = \phi_m$ and $v = \phi_n$, then

$$\int_a^b [\phi_m L(\phi_n) - \phi_n L(\phi_m)]\,dx = p(x)\left(\phi_m\frac{d\phi_n}{dx} - \phi_n\frac{d\phi_m}{dx}\right)\Big|_a^b$$

From (P), $$L(\phi_i) = -\lambda_i\sigma(x)\phi_i, \quad i = n, m$$

so $$\phi_m L(\phi_n) = \phi_m(-\lambda_n\sigma(x)\phi_n)$$
$$\phi_n(L(\phi_m)) = \phi_n(-\lambda_m\sigma(x)\phi_m)$$
$$\int_a^b -\lambda_n\sigma\phi_m\phi_n - (-\lambda_m\sigma\phi_n\phi_m) = (\lambda_m - \lambda_n)\int_a^b \phi_n\phi_m\sigma\,dx.$$

So

$$(\lambda_m - \lambda_n)\int_a^b \phi_n \phi_m \sigma\, dx = p(x)\left(\phi_m \frac{d\phi_n}{dx} - \phi_n \frac{d\phi_m}{dx}\right)\Big|_a^b$$

Since ϕ_n, ϕ_m are eigenfunctions the boundary term vanishes because they satisfy homogeneous boundary conditions. This implies

$$(\lambda_m - \lambda_n)\int_a^b \phi_n \phi_m \sigma\, dx = 0 \qquad (*).$$

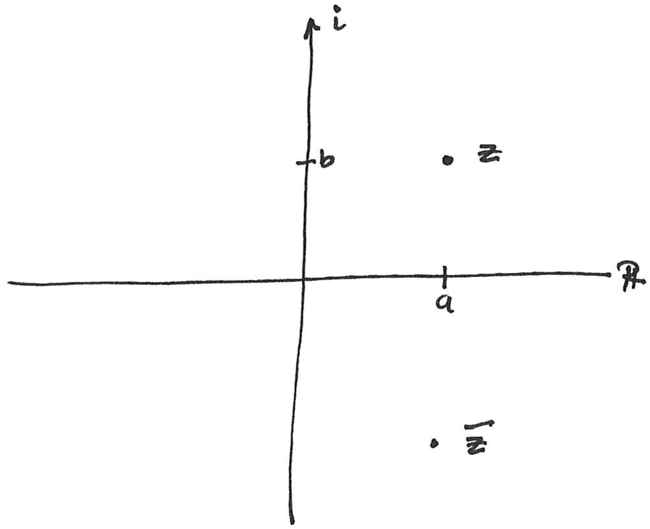
therefore if $\lambda_m \neq \lambda_n$ it follows that

$$\int_a^b \phi_n \phi_m \sigma\, dx = 0. \qquad \square$$

<u>Eigenvalues are real</u>

complex numbers $z = a + bi$, $\bar{z} = a - bi$

If $z \in \mathbb{R}$, $\bar{z} = z$



In general

$$\bar{z}z = (a+bi)(a-bi)$$
$$a^2 - abi + bia - b^2 i^2$$
$$a^2 + b^2 = |z|^2$$

Suppose λ is a complex eigenvalue with a corresponding (possibly complex) eigenfunction $\phi(x)$:

$$L(\phi) + \lambda \sigma \phi = 0.$$

Taking the complex on both sides

$$\overline{L(\phi)} + \bar{\lambda} \sigma \bar{\phi} = 0$$

$$L(\bar{\phi}) + \bar{\lambda} \sigma \bar{\phi} = 0 \qquad * \text{ if } \phi \text{ satisfies boundary conditions}$$

with real coefficients, then $\bar{\phi}$ satisfies the same boundary conditions.

so $\bar{\Phi}$ satisfies the (SLP) with eigenvalue $\bar{\lambda}$.

i.e if $\lambda \in \mathbb{C}$ is an eigenvalue corresponding to ϕ, then $\bar{\lambda}$ is an eigenvalue corresponding to $\bar{\Phi}$.

$$L(\bar{\Phi}) + \lambda \sigma \phi = 0 \quad \cdots \quad (1) \quad \to \phi$$

$$L(\bar{\Phi}) + \bar{\lambda} \sigma \bar{\Phi} = 0 \quad \cdots \quad (2)$$

From Green's identity

$$\int_a^b \left[\bar{\Phi} L(\phi) - \phi L(\bar{\Phi})\right] dx = 0$$

From (1) $\bar{\Phi} L(\phi) + \lambda \sigma \bar{\Phi} \phi = 0 \Rightarrow \bar{\Phi} L(\phi) = -\lambda \sigma \bar{\Phi} \phi$

(2) $\phi L(\bar{\Phi}) + \bar{\lambda} \sigma \phi \bar{\Phi} = 0 \Rightarrow \phi L(\bar{\Phi}) = -\bar{\lambda} \sigma \bar{\Phi} \phi$.

so that

$$(\lambda - \bar{\lambda}) \int_a^b \phi \bar{\Phi} \sigma \, dx = 0$$

Since $\phi \bar{\Phi} = |\phi|^2 \geq 0$ and $\sigma \geq 0$, $\int_a^b \phi \bar{\Phi} \sigma dx > 0$ so it must
(ϕ cannot be zero)

be the case that $\lambda = \bar{\lambda}$ so λ is real.

Uniqueness (hw)

Rayliegh Quotient

For the (SL)- ode

$$\frac{d}{dx}\left[p(x) \frac{d\phi}{dx}\right] + q(x) \phi + \lambda \sigma(x) \phi = 0$$

$$\lambda = \frac{-p \phi \frac{d\phi}{dx}\Big|_a^b + \int_a^b \left[p \left(\frac{d\phi}{dx}\right)^2 - q \phi^2\right] dx}{\int_a^b \phi^2 \sigma \, dx}$$

Positive eigenvalues ($\lambda \geq 0$)

HEAT FLOW : $\lambda > 0$ corresponds to solutions that decay exponentially in time $\left(\frac{dh}{dt} = -\lambda h\right)$ is the time dependent problem.

In VIBRATIONS

$$\frac{d^2h}{dt^2} = -\lambda h , \quad \lambda > 0 \quad \text{oscillations} .$$

Rayleigh quotient proves that $\lambda \geq 0$ if

(a) $-p\,\phi \frac{d\phi}{dx}\Big|_a^b \geq 0$ and

(b) $q \leq 0$.

(a) $\Rightarrow$ Homogeneous boundary conditions, $\phi \equiv 0$ and $\frac{d\phi}{dx} = 0$ $\Rightarrow$ zero boundary term

$\Rightarrow$ Newton's Law of cooling $\frac{d\phi}{dx} = h\phi$, $h > 0$ $\Rightarrow$ positive contribution

$\Rightarrow$ Periodic conditions also cause a positive contribution.

(b) $q(x)$ in physical problems corresponds to

(1) Endothermic reactions

(2) Restoring force in vibrations.

MINIMIZATION PRINCIPLE

* Rayleigh quotient cannot be used to compute λ since ϕ is unknown.

FACT

The minimum value of the Rayleigh quotient for all continuous functions satisfying boundary conditions (not necessarily the ODE) is the lowest eigenvalue

$$\lambda_1 = \min \frac{-p\,u \frac{du}{dx}\Big|_a^b + \int_a^b \left[p \left(\frac{du}{dx}\right)^2 - q u^2 \right] dx}{\int_a^b u^2 \sigma \, dx} \qquad (*)$$

The minimum is attained for $u(x) = \phi_1(x)$.

BOUNDING THE LOWEST EIGEN-VALUE

Motivation

$$u(x,t) = \phi_n(t)\, e^{-\lambda_n t} \Rightarrow \sum_{n=1}^{\infty} a_n \phi_n(x)\, e^{-\lambda_n t}$$

which can be approximated as

$$u(x,t) = a_1 \phi_1(x)\, e^{-\lambda t}$$

Since we do not know how to minimize over all functions, we cannot use (*).

Let u_T be any continuous function satisfying the boundary conditions (TRIAL FUNCTION)

$$\lambda_1 \leq RQ[u_T] = \frac{-p u_T \frac{du_T}{dx}\Big|_a^b + \int_a^b \left[p\left(\frac{du_T}{dx}\right)^2 - q u_T^2 \right] dx}{\int_a^b u_T^2 \sigma\, dx}$$

Example

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0$$

$$\phi(0) = 0$$

$$\phi(1) = 0$$

$$\boxed{\begin{aligned} p(x) &= 1 \\ q(x) &= 0 \\ \sigma(x) &= 0. \end{aligned}}$$

TRUE SOLUTION

$$\phi_1(x) = \sin(\pi x)$$

$$\lambda_1 = \left(\frac{n\pi}{L}\right)^2 \quad \lambda = \pi^2.$$

Choosing trial functions (Continuous)

① They should satisfy boundary conditions

② No zeros in the interior because ϕ_1 has no zeros.

LINEAR TRIAL FUNCTION

$$\text{(i)}\quad u_T = \begin{cases} x, & x < \frac{1}{2} \\ 1-x, & x > \frac{1}{2}. \end{cases}$$

$$\lambda_1 \leq \frac{\int_0^1 \left(\frac{du_T}{dx}\right)^2 dx}{\int_0^1 u_T^2\, dx} = \frac{\int_0^{\frac{1}{2}} 1^2 dx + \int_{\frac{1}{2}}^1 1\, dx}{\int_0^{\frac{1}{2}} x^2 dx + \int_{\frac{1}{2}}^1 (1-x)^2 dx} = \frac{1}{\frac{1}{24} + \frac{1}{12}} = 12.$$

$$\text{Relative error} = \frac{12 - \pi^2}{\pi^2} \cdot 100 \cong 21\%$$

Quadratic trial function

$$u_T = x(1-x) = x - x^2.$$

$$\lambda_1 \le \frac{\int_0^1 (1-2x)^2\,dx}{\int_0^1 (x-x^2)^2\,dx} = 10$$

$$\frac{10-\pi^2}{\pi^2} \cdot 100 \simeq 1.3\%$$

If we choose $u_T = \sin(\pi x)$, we get π^2 (the true eigenvalue).

Application of Rayleigh Quotient : Vibrations of a uniform string

Assumptions

constant tension $- T_0$

variable mass density $- \rho(x)$

no sources $- Q = 0$.

zero displacement on both ends.

$$\rho \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0$$

Boundary conditions

$$u(0,t) = 0$$
$$u(L,t) = 0$$

Initial conditions

$$u(x,0) = f(x)$$
$$\frac{\partial u}{\partial t}(x,0) = g(x)$$

From separation of variables

$$u(x,t) = \phi(x)\, h(t)$$

$$\frac{d^2h}{dt^2} = -\lambda h \qquad (*)$$

$$T_0 \frac{d^2\phi}{dx^2} + \lambda \rho(x)\phi = 0$$

$$\phi(0) = \phi(L) = 0.$$

Skip

Physically, we expect $\lambda > 0$ (due to oscillations)

Rayleigh Quotient
$$\lambda = \frac{T_0 \int_0^L \left(\frac{d\phi}{dx}\right)^2 dx}{\int_0^L \phi^2 \rho(x)\, dx}$$

$P = T_0$

$\sigma(x) = \rho(x)$

It is possible for $\lambda = 0$, when $\phi = \text{constant}$. $\lambda \geq 0$.

This assures us that the solution of $(*)$ is a linear combination of

$\sin\sqrt{\lambda}t$ and $\cos\sqrt{\lambda}t$.

Product solutions are $\sin(\sqrt{\lambda_n})\,\phi_n(t)$ and $\cos\sqrt{\lambda_n}t\ \phi_n(t)$.

so
$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin\sqrt{\lambda_n}t\ \phi_n(x) + \sum_{n=1}^{\infty} b_n \cos(\sqrt{\lambda_n}t)\ \phi_n(x)$$

where a_n and b_n ~~can~~ are computed from the initial position and velocity

<u>Information about the lowest eigenvalue λ_1</u>

$$\lambda_1 = \min \frac{T_0 \int_0^L \left(\frac{du}{dx}\right)^2 dx}{\int_0^L u^2 \rho(x)\, dx}, \quad \rho(x) = \text{mass density}$$

Suppose $0 < \rho_{min} \leq \rho(x) \leq \rho_{max}$. Then for any $u(x)$, it follows that

$$\rho_{min} \int_0^L u^2 dx \leq \int_0^L u^2 \rho(x)\, dx \leq \rho_{max} \int_0^L u^2 dx$$

So that

$$\frac{T_0}{\rho_{max}} \min \frac{\int_0^L \left(\frac{du}{dx}\right)^2 dx}{\int_0^L u^2 dx} \leq \lambda_1 \leq \frac{T_0}{\rho_{min}} \frac{\int_0^L \left(\frac{du}{dx}\right)^2 dx}{\int_0^L u^2 dx}$$

Regconize that

$$\min \frac{\int_0^L \left(\frac{du}{dx}\right)^2 dx}{\int_0^L u^2 dx} \quad \text{subject to} \quad u(0) = u(L) = 0$$

is the lowest eigenvalue of the BVP

$$\frac{d^2\phi}{dx^2} + \lambda^* \phi = 0$$

$$\phi(0) = 0 \quad \text{and} \quad \phi(L) = 0$$

where $\lambda_n^* = \left(\frac{n\pi}{L}\right)^2$ so $\lambda_1^* = \frac{\pi^2}{L^2}$

by the minimization principle

$$\lambda_1^* = \min \frac{\int_0^L \left(\frac{du}{dx}\right)^2 dx}{\int_0^L u^2 dx}$$

so

$$\frac{T_0}{\rho_{max}} \left(\frac{\pi}{L}\right)^2 \leq \lambda_1 \leq \frac{T_0}{\rho_{min}} \left(\frac{\pi}{L}\right)^2$$

$$\Rightarrow \frac{\pi}{L} \sqrt{\frac{T_0}{\rho_{max}}} \leq \sqrt{\lambda_1} \leq \frac{\pi}{L} \sqrt{\frac{T_0}{\rho_{min}}}$$

Lowest frequency of oscillation of a variable string lies between the lowest frequencies of the max and ~~max~~ min densities.