FOURIER SERIES

$$(*) \quad \boxed{f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \quad -L \le x \le L}$$

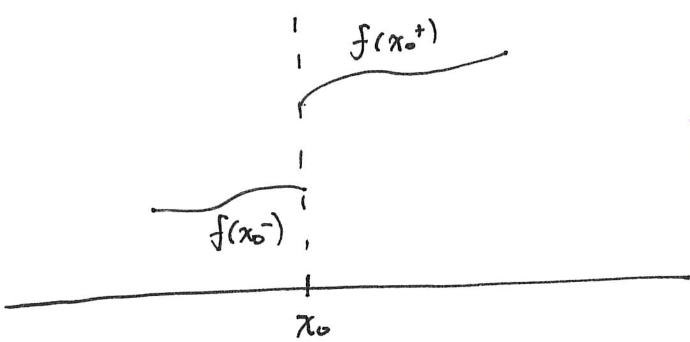
* does the series converge?

** does the series converge to $f(x)$?

*** We study the general case, the sine or cosine series are just special cases of (*).

We focus on piecewise-smooth functions $\Rightarrow$ the interval on which the function is defined can be broken up into sub-intervals on which $f(x)$ is continuous & df/dx is also continuous.

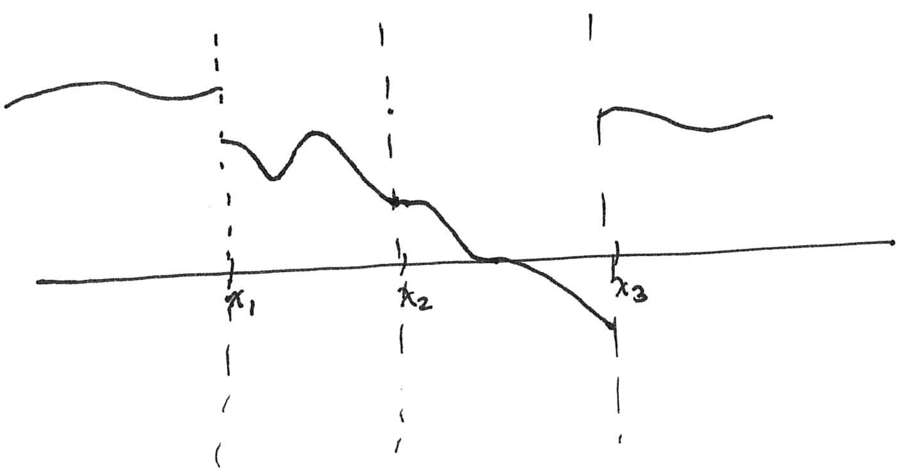
f may have a finite number of jump discontinuities



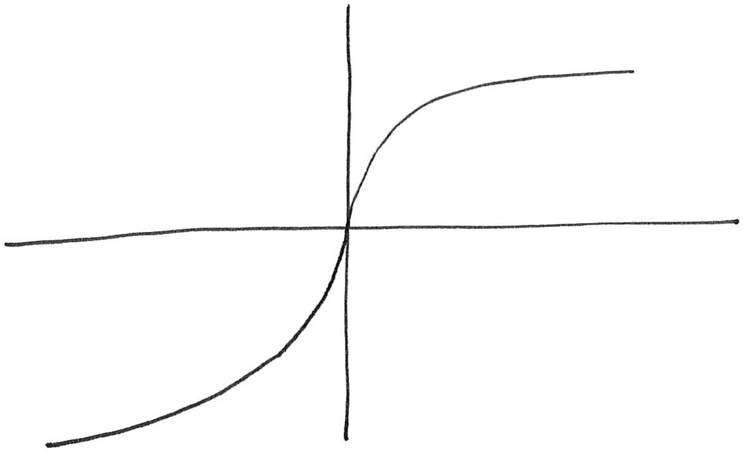
$[f(x_0^-)] \neq [f(x_0^+)]$.

$\lim_{x \to x_0^-} f(x) \neq \lim_{x \to x_0^+} f(x)$. (left & right limits exist But are not equal)

Example of piecewise smooth



$f(x) = x^{\frac{1}{3}}$

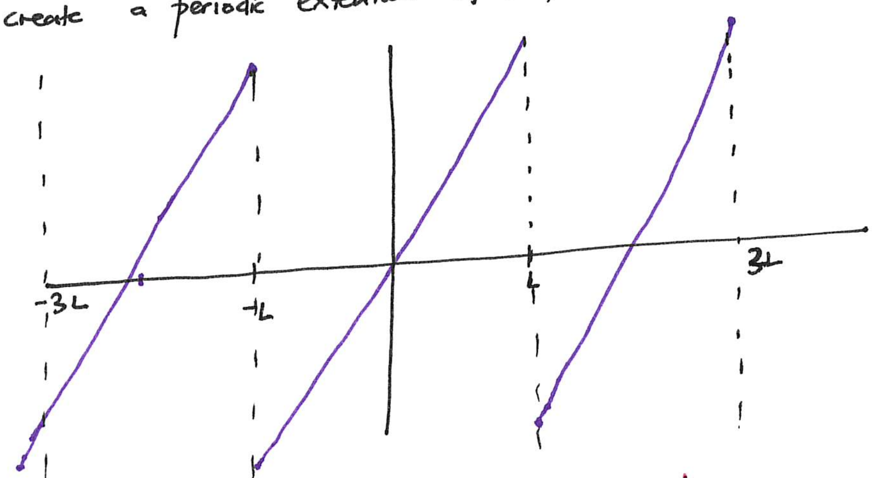


$f(x) = x^{\frac{1}{3}}$ is not piecewise smooth on any interval that contains $x=0$.
because $f'(x) = \frac{1}{3}x^{-2/3}$ is discontinuous @ $x=0$.

Periodic extensions

The series of $f(x)$ on $-L \le x \le L$ is periodic on $2L$, but $f(x)$ need not be periodic with period $2L$.

We can create a periodic extention of $f(x)$



periodic extension of $f(x) = \frac{3}{2}x$

Convergence Theorem (FOURIER'S THEOREM)

If $f(x)$ is piecewise smooth on $-L \le x \le L$, then the Fourier series of $f(x)$ converges to

1. The periodic extension of $f(x)$, where the periodic extension is continuous
2. To the ~~avera~~ average of the limits where $f(x)$ has a jump discontinuity

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

Remarks

1. The series converges to $f(x)$ for $-L < x < L$ provided $f(x)$ is cont. piecewise smooth.
2. Converges to the average of $f(-L)$ and $f(L)$ at the endpoints

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Notation

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$f(x)$ has the given fourier series on $-L \le x \le L$

* Just because the coefficients a_n, b_n exist does not mean the series converges.

FOURIER SINE SERIES

Odd functions

$$f(-x) = -f(x)$$ (integral over a symmetric interval $=0$)

Fourier series of an odd function

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = 0$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = 0$$, because $f(x)\cos\left(\frac{n\pi x}{L}\right)$ is odd.

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$ if $f(x)$ is odd $\quad (*)$

where $$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \;\underset{\uparrow}{=}\; \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

because $f(x)\sin\left(\frac{n\pi x}{L}\right)$ is even. (product of 2 odd functions).

Recall the temperature in 1D, $0<x<L$ with zero-temperature ends

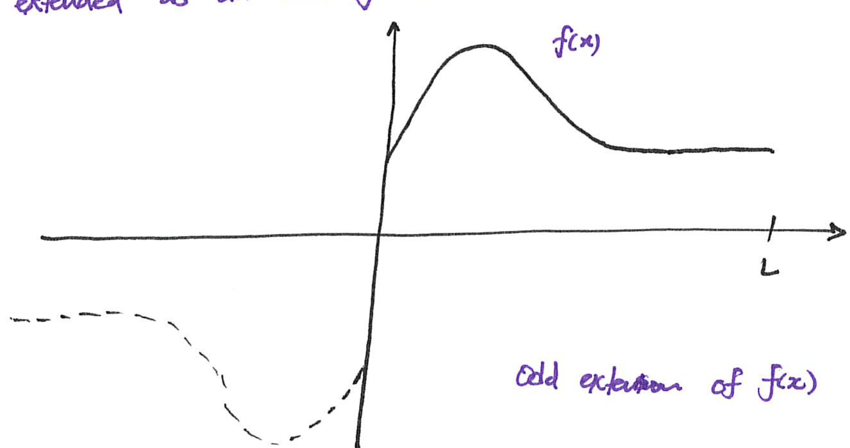
$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt}$$

where the initial condition is satisfied if

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \qquad (**)$$

$(**)$ takes the form of $(*)$, however in $(**)$ $f(x)$ is only defined on $0 \le x \le L$

$f(x)$ may not be odd, however, if $f(x)$ is defined on $0 \le x \le L$, $f(x)$ can be extended as an odd function



Odd extension of $f(x)$

$$\text{odd extension of } f(x) \sim \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right), \quad -L \le x \le L.$$

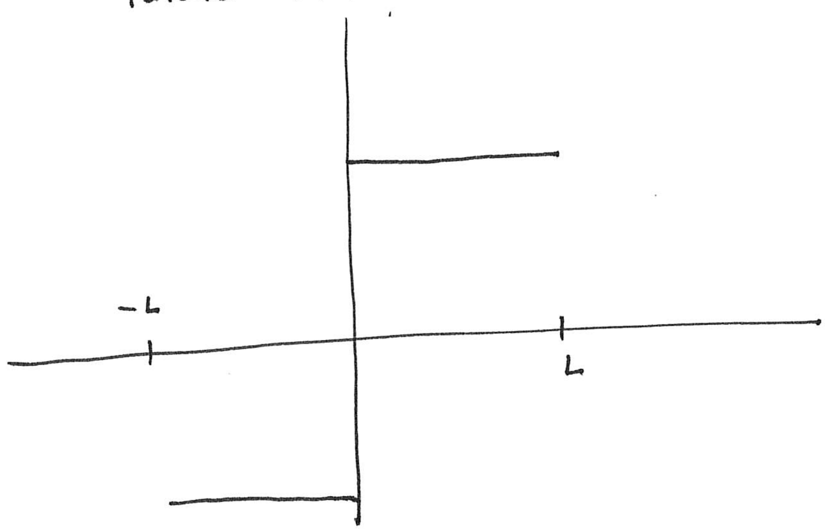
In the 1D heat example case

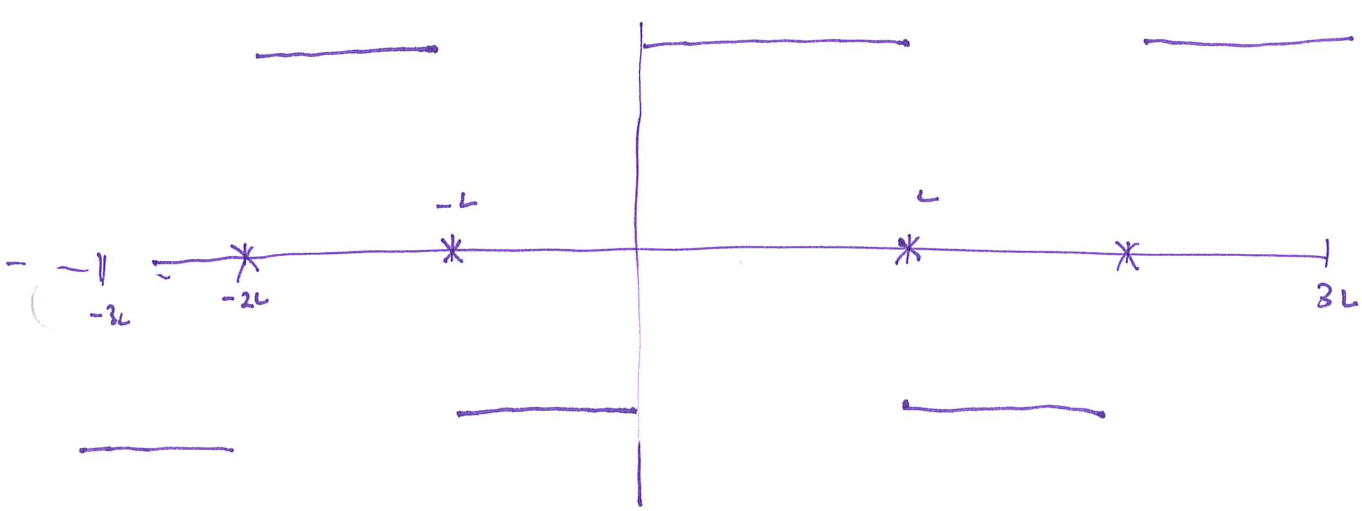
$$f(x) \sim \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right), \quad 0 < x < L.$$

Sketching the Fourier sine series.

1. Sketch $f(x)$ $(0 < x < L)$
2. Sketch the periodic extension
3. Extend as a periodic function to period $2L$.
4. Mark x at the average points where the odd periodic extension has a jump discontinuity

Periodic extension of $f(x) = 100$.





Physical example

$$\boxed{\begin{array}{l} \dfrac{\partial u}{\partial t} = k \dfrac{\partial^2 u}{\partial x^2} \\ u(0,t) = 0 \\ u(L,t) = 0 \\ u(x,0) = 100 \end{array}} \quad \text{on } 0<x<L, \; t>0$$

From separation of variables we got

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt}$$

When the initial conditions are true if

$$\boxed{100 = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right), \quad 0<x<L.} \quad *$$

(*) is the Fourier series of 100, however evaluating the series @ $x=0$ and $x=L$ yields 0!

(**) The physical problem has a discontinuity @ $t=0$, ($x=0$ & $x=L$) between the initial condition and the boundary condition.

i.e. Initial condition prescribes ~~u(x,~~ temperature $= 100$ ~~@~~ as $x \to 0$ but the boundary condition imposes 0

(***) This is situation arises because we "instantaneously" transport the heated rod to a 0° bath at the end points.

Built in discontinuity!

Fourier series of 100 captures this ~~Also proper~~ property i.e. it equals 100 for $0<x<L$ and it equals zero at the endpoints.

Gibbs Phenomenon

As we increase n the size of the overshoots does not decrease, instead it converges to $\cong$ 9% of the size of the jump discontinuity.

$\cong$ 118.

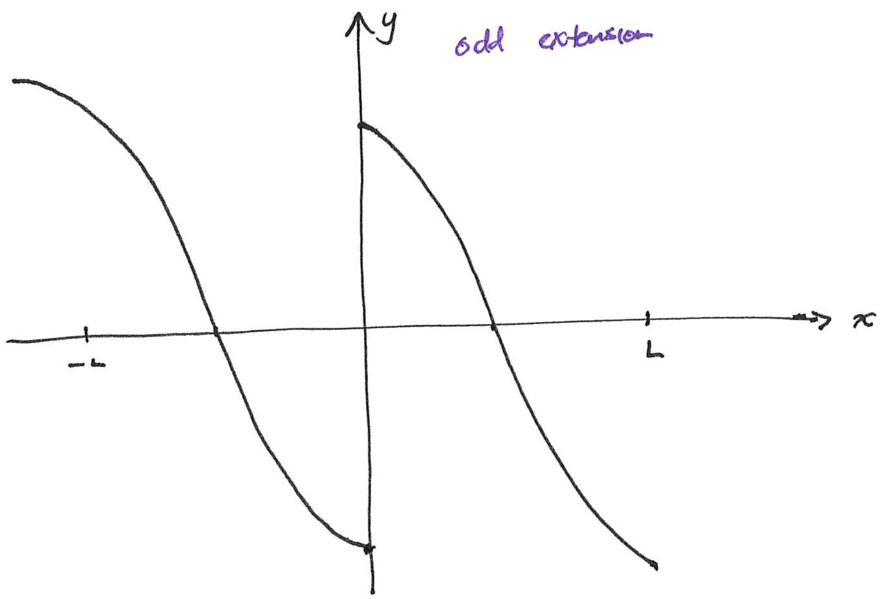
Other examples of Fourier sine series

e.g. $f(x) = \cos\left(\frac{n\pi x}{L}\right)$

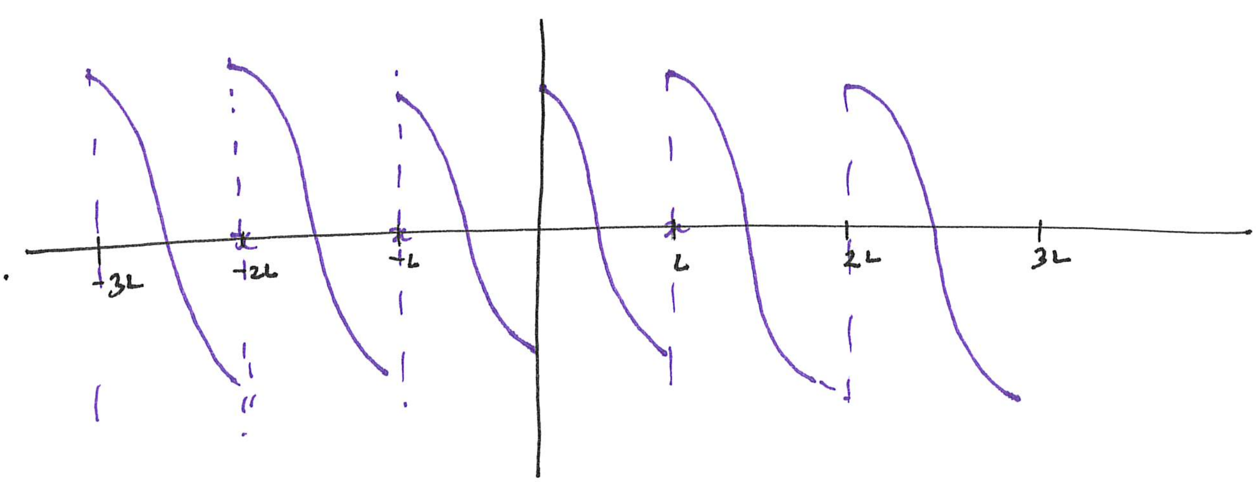
It may seem strange to consider a Fourier sine series expansion of $\cos\left(\frac{n\pi x}{L}\right)$ but this may arise in the context of solving

$$\phi'' = -\lambda\phi, \quad \phi(0) = 0$$

the heat equation with dirichlet b.c. on $0 < x < L$.



Then the fourier series of $f(x) = \cos\left(\frac{n\pi x}{L}\right)$ is



$$\cos\left(\frac{n\pi x}{L}\right) \sim \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right), \quad 0 \le x \le L$$

Where

$$B_n = \frac{2}{L}\int_0^L \cos\left(\frac{\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \begin{cases} 0, & n \text{ odd} \\ \dfrac{4n}{\pi(n^2-1)}, & n \text{ even.} \end{cases}$$

<u>Remark</u>

$$\cos\left(\frac{n\pi x}{L}\right) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \qquad 0 < x < L$$

@ $x = 0$ and $x = L$, the series converges to zero.

Fourier Cosine Series . $f(-x) = f(x)$ (even functions)

In the case of even functions, the sine coefficients

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right) = 0$$

so the fourier series for an even function is

$$f(x) \sim \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

with coefficients

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{L}\int_{0}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \quad , n \geq 1.$$

In general, if $f(x)$ is not an even function

$$\boxed{\text{even extension of } f(x) \sim \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \qquad -L \leq x \leq L.}$$

Fourier Cosine Series.

Representing $f(x)$ by both sine and cosine series

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)$$

In general we can write

$$f(x) = \underbrace{\frac{1}{2}\left[f(x) + f(-x)\right]}_{\text{even } (f_e(x))} + \underbrace{\frac{1}{2}\left[f(x) - f(-x)\right]}_{\text{odd. } f_o(x)}$$

So ~~say~~ In general $f(x)$ = fourier sine series of $f_o(x)$ + fourier cosine series of $f_e(x)$.

Fourier's Theorem

If $f(x)$ is piecewise smooth on $-L \leq x \leq L$, then the fourier series of $f(x)$ converges to

1. The periodic extension of $f(x)$, where the periodic extension is continuous
2. The average of the limits where $f(x)$ has a jump discontinuity

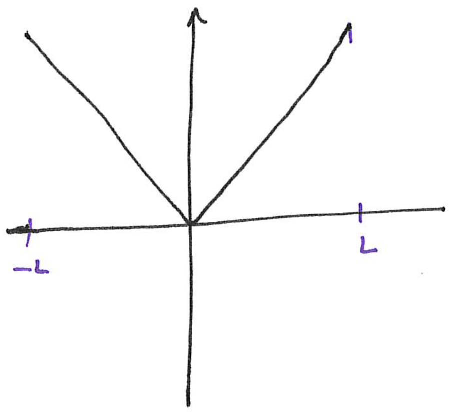
* A piecewise continuous function, $f(x)$ on an interval $[a, b]$ is piecewise smooth if $f'(x)$ is piecewise continuous.

Continuity of Fourier Series

For a piecewise smooth function $f(x)$, the Fourier series is continuous and converges to $f(x)$ on $-L \le x \le L$ if and only if $f(x)$ is continuous and $f(-L) = f(L)$.

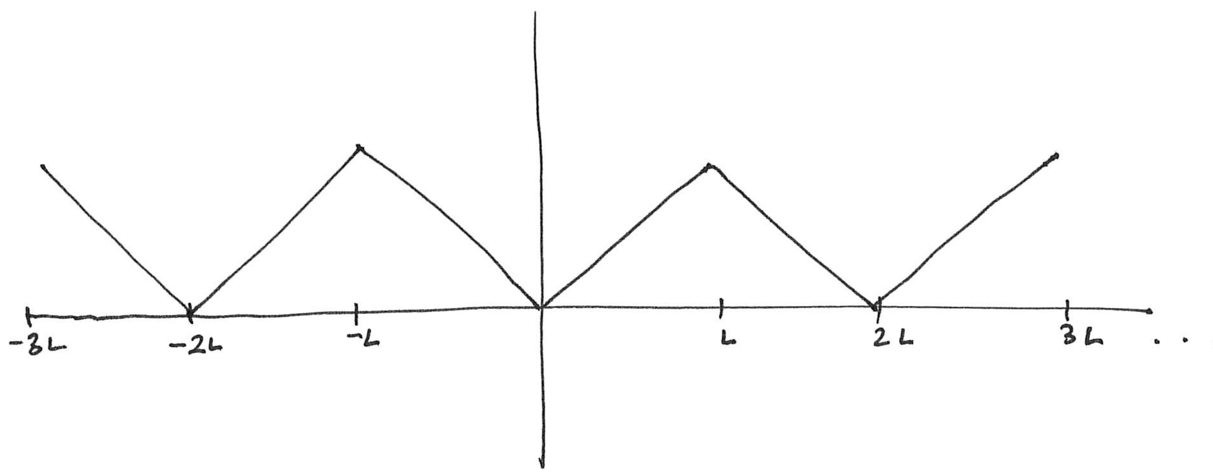
Example

$f(x) = x$ on $0, L$



Even though $f(x)=x$ is odd, if we start with $f(x)=x$ on $[0,L]$ we can create an even extension of $f(x)$.

then the Fourier series of the even extension of $f(x)=x$ is



$$f(x) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \quad , 0 \le x \le L.$$

where the coefficients are given by $A_0 = \frac{1}{L}\int_0^L x\,dx = \frac{L}{2}$

$$A_n = \frac{2}{L}\int_0^L x\cos\left(\frac{n\pi x}{L}\right)dx$$

Integrate by parts $u = x$
$dv = \cos\left(\frac{n\pi x}{L}\right)$

$$= \frac{2L}{(n\pi)^2}\left(\cos(n\pi) - 1\right)$$

Let $f(x)$ be piecewise smooth, then the Fourier cosine series is continuous and converges to $f(x)$ for $0 \le x \le L$ if and only if $f(x)$ is continuous.

Fourier sine series

For piecewise smooth functions, the Fourier series of $f(x)$ is continuous and converges to $f(x)$ for $0 \le x \le L$ if and only if $f(x)$ is continuous and both $f(0) = 0$ and $f(L) = 0$.

Remark

No additional conditions are necessary for the Fourier cosine series to be continuous (besides ~~piecewise~~ continuity of $f(x)$) because the even extension will be continuous on $-L \le x \le L$ and hence the periodic extension will be continuous at endpoints.

Term by Term differentiation

We have established that

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt} \qquad (*)$$

solves $$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

$$u(0,t) = 0 \quad , u(L,t) = 0 \quad , \quad u(x,0) = f(x)$$

To check this solution we can plug in (*) into the pde and differentiate term by term.

LHS $$\frac{\partial u}{\partial t} = -\sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right)^2 k \, B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt}$$

RHS $$\frac{\partial^2 u}{\partial x^2} = -\sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right)^2 B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt}.$$

Seems easy enough.... However, unfortunately infinite series <u>cannot</u> always be differentiated term by term (even convergent ones!)

i.e $$\frac{d}{dx}\left(\sum_{n=1}^{\infty} C_n u_n\right) = \sum_{n=1}^{\infty} C_n \frac{du_n}{dx}$$

is not always true.

Counter - example

Fourier series of sine $f(x) = x$

$$x = 2\sum_{n=1}^{\infty} \frac{L}{n\pi}(-1)^{n+1} \sin\left(\frac{n\pi x}{L}\right) \qquad \text{on } 0<x<L$$

<u>note</u>: coefficients

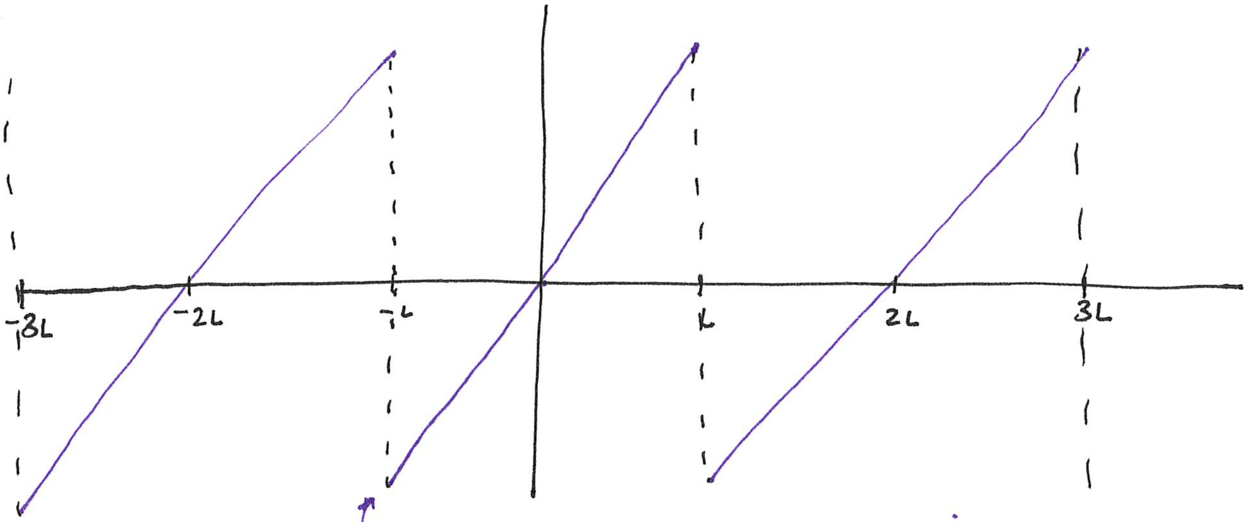
$$a_n = \frac{2}{L}\int_0^L x \sin\left(\frac{n\pi x}{L}\right) dx$$

* integrate by parts!

If we differentiate term by term we get

$$2\sum_{n=1}^{\infty} (-1)^{n+1} \cos\left(\frac{n\pi x}{L}\right) \neq 1.$$

The problem arises due to the jump discontinuity in the Fourier series of $f(x) = x$ (the fourier sine series, that is).



differentation does not hold at the jumps

FACT I

A continuous fourier series can be differentiated term by term if $f'(x)$ is piecewise smooth. i.e $f'(x)$ has a fourier series!

Alternatively

If $f(x)$ is piecewise smooth, then the Fourier series of a continuous function $f(x)$ can be differentiated term by term provided $f(-L) = f(L)$

Fourier Cosine Series

If $f'(x)$ is piecewise smooth, then a continuous Fourier cosine series of $f(x)$ can be differentiated term by term. / *If $f'(x)$ is piecewise smooth, then the Fourier cosine series of a continuous function, $f(x)$ can be differentiated term by term.

In summary

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \quad , \quad 0 \leq x \leq L$$

$$f'(x) \sim -\sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right) A_n \sin\left(\frac{n\pi x}{L}\right)$$

If $f'(x)$ is continuous, the series converges to $f'(x)$ otherwise to the average at jumps

Fourier Sine Series

If $f'(x)$ is piecewise smooth, then the Fourier sine series of a continuous function $f(x)$ can be differentiated term by term if and only if $f(0) = 0$ and $f(L) = 0$.

Proof We assume that $f'(x)$ is piecewise smooth and $f(0) = 0 = f(L)$.

Suppose
$$f(x) \sim \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \qquad (*)$$

[Equality holds if $f(0) = 0 = f(L)$]

where
$$B_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

Since ~~If~~ $f'(x)$ is piecewise smooth, then $f'(x)$ has a Fourier cosine series

$$f'(x) \sim A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)$$

where
$$A_0 = \frac{1}{L}\int_0^L f'(x)\,dx = \frac{1}{L}\left(f(L) - f(0)\right)$$

$$A_n = \frac{2}{L}\int_0^L f'(x) \cos\left(\frac{n\pi x}{L}\right) dx = \frac{2}{L}\left[f(x)\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L + \frac{n\pi}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) dx\right]$$

$$\boxed{\begin{array}{ll} dv = f'(x)dx & u = \cos\left(\frac{n\pi x}{L}\right) \\ v = f(x) & \frac{du}{dx} = -\frac{n\pi}{L}\sin\left(\frac{n\pi x}{L}\right) \end{array}}$$

valid if $f(x)$ is continuous.

so for $n \geq 1$,

$$A_n = \frac{n\pi}{L} B_n + \frac{2}{L}\left[f(L)(-1)^n - f(0)\right] \qquad (**)$$

We conclude by comparing the Fourier cosine coefficients $(**)$ that the Fourier sine series $(*)$ can be differentiated ~~if~~ term by term if $f(L) = f(0) = 0$.

If $f'(x)$ is piecewise smooth, then the Fourier series of a continuous function $f(x)$,

$$f(x) \sim \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

cannot, in general be differentiated. However

$$f'(x) \sim \frac{1}{L}\left[f(L)-f(0)\right] + \sum_{n=1}^{\infty}\left[\frac{n\pi}{L} B_n + \frac{2}{L}\left((-1)^n f(L) - f(0)\right)\right] \cos\left(\frac{n\pi x}{L}\right)$$

Method of eigenfunction expansion

$$\boxed{\begin{aligned} &\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad 0<x<L, \quad t>0 \\ &u(0,t) = u(L,t) = 0, \quad u(x,0) = f(x) \end{aligned}} \qquad (P)$$

Assuming that $u(x,t)$, $\frac{\partial u}{\partial t}$, $\frac{\partial u}{\partial x}$, $\frac{\partial^2 u}{\partial x^2}$ are piecewise smooth & continuous on $[0,L]$, we expand $u(x,t)$ in terms of the eigenfunctions

$$u(x,t) \sim \sum_{n=1}^{\infty} B_n(t) \sin\left(\frac{n\pi x}{L}\right) \qquad (*)$$

The initial condition is satisfied if

$$f(x) \sim \sum_{n=1}^{\infty} B_n(0) \sin\left(\frac{n\pi x}{L}\right)$$

so $B_n(0) = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$

If $u(x,t)$ is continuous, the Fourier sine! series can be differentiated term by term provided $u(0,t)=0$ and $u(L,t)=0$. (these are the boundary conditions!).

$$\text{so} \quad \frac{\partial u}{\partial x} \sim \sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right) B_n(t)\cos\left(\frac{n\pi x}{L}\right) \qquad (\text{from } *)$$

Since $\frac{\partial u}{\partial x}$ is also continuous, the Fourier cosine series can be differentiated term by term so (Fourier cosine series has no restrictions besides continuity)

$$\frac{\partial^2 u}{\partial x^2} \sim -\sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right)^2 B_n(t)\sin\left(\frac{n\pi x}{L}\right)$$

Term by Term differentiation w.r.t t

Thm

The Fourier series of a continuous function, $u(x,t)$

$$u(x,t) = a_0(t) + \sum_{n=1}^{\infty}\left[a_n(t)\cos\left(\frac{n\pi x}{L}\right) + b_n(t)\sin\left(\frac{n\pi x}{L}\right)\right]$$

can be differentiated term by term if $\frac{\partial u}{\partial t}$ is piecewise smooth

Infact,

$$\frac{\partial}{\partial t} u(x,t) = a_0'(t) + \sum_{n=1}^{\infty} \left(a_n'(t) \cos\left(\frac{n\pi x}{L}\right) + b_n'(t) \sin\left(\frac{n\pi x}{L}\right)\right)$$

If $\frac{\partial u}{\partial t}$ is piecewise smooth.

so in our case

$$\frac{\partial u}{\partial t} \sim \sum_{n=1}^{\infty} \frac{d}{dt} B_n(t) \sin\left(\frac{n\pi x}{L}\right)$$

Plugging in the series into the PDE

$$\sum_{n=1}^{\infty} \frac{d}{dt} B_n(t) \sin\left(\frac{n\pi x}{L}\right) = -k \sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right)^2 B_n \sin\left(\frac{n\pi x}{L}\right)$$

so the series solves the PDE provided

$$\frac{d}{dt} B_n(t) = -k \left(\frac{n\pi}{L}\right)^2 B_n(t)$$

Solving yeilds

$$B_n(t) = B_n(0)\, e^{-\left(\frac{n\pi}{L}\right)^2 kt}$$

so

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} B_n(0)\, e^{-\left(\frac{n\pi}{L}\right)^2 kt} \sin\left(\frac{n\pi x}{L}\right)}$$ is the solution.

<u>Extension to non-homogeneous problems</u>

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + e^{-t} + e^{-2t} \cos\left(\frac{3\pi x}{L}\right)$$

subject to

$$\frac{\partial u}{\partial x}(0,t) = 0, \quad \frac{\partial u}{\partial x}(L,t) = 0 \quad \text{and}$$

$$u(x,0) = f(x).$$

Reminder

Method of integrating factors

$$y' + ky = q(t)$$

Integrating factor is e^{kt}, multiply on both sides

$$\boxed{e^{kt}(y' + ky)} = q(t)e^{kt}$$

Notice that $(y(t)e^{kt})' = y'e^{kt} + ke^{kt}y = e^{kt}(y' + ky)$

so

$$(y(t)e^{kt})' = q(t)e^{kt}$$

so we can integrate on both sides . . .

$$y(t)e^{kt} = \int_0^t q(w)e^{kw}\,dw + y(0)$$

$$\Rightarrow \quad y(t) = e^{-kt}\left(\int_0^t q(w)e^{kw}\,dw + y(0)\right)$$

Assuming u, u_t, u_x, u_{xx} are piecewise smooth and continuous we expand $u(x,t)$ interms of the eigen functions

$$u(x,t) \sim \sum_{n=0}^{\infty} A_n(t) \cos\left(\frac{n\pi x}{L}\right)$$

u is continuous therefore the Fourier cosine series can be differentiated term by term

$$\frac{\partial u}{\partial x} \sim -\sum_{n=0}^{\infty} A_n(t) \left(\frac{n\pi}{L}\right) \sin\left(\frac{n\pi x}{L}\right)$$

This can be differentiated term by term because $\frac{\partial u}{\partial x}$ is continuous and

$$\frac{\partial u}{\partial x}(0) = 0 \quad \text{and} \quad \frac{\partial u}{\partial x}(L) = 0$$

$$\frac{\partial^2 u}{\partial x^2} \sim -\sum_{n=0}^{\infty} A_n(t) \left(\frac{n\pi}{L}\right)^2 \cos\left(\frac{n\pi x}{L}\right).$$

Plug into PDE

$$\sum_{n=0}^{\infty} \left[\frac{d}{dt} A_n(t) + k\left(\frac{n\pi}{L}\right)^2 A_n\right] \cos\left(\frac{n\pi x}{L}\right) = e^{-t} + e^{-2t} \cos\left(\frac{3\pi x}{L}\right)$$

Notice that the Rhs is a cosine series with 2 non-zero terms.

$$\frac{dA_n}{dt} + k\left(\frac{n\pi}{L}\right)^2 A_n = \begin{cases} e^{-t}, & n=0 \\ e^{-2t}, & n=3 \\ 0, & \text{otherwise.} \end{cases}$$

The initial conditions are

$$u(x,0) = A_0(0) = \frac{1}{L}\int_0^L f(x)\,dx$$

$n \neq 0$

$$A_n(0) = \frac{2}{L}\int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx.$$

$n=0$

$$\frac{d}{dt} A_0(t) = e^{-t}$$

Integrating both sides

$$A_0(t) = \int_0^t e^{-w}\,dw + c \qquad \text{by} \quad FTC^*$$

Infact

$$A_0(t) = \int_0^t e^{-w}\,dw + A_0(0) \qquad \text{Integrating} \quad -e^{-w}\Big|_0^t = -e^{-t} - (-1)$$

$$1 - e^{-t}$$

$$\boxed{A_0(t) = A_0(0) + 1 - e^{-t}}$$

$n=3$

$$\frac{d}{dt} A_3(t) + k\left(\frac{3\pi}{L}\right)^2 A_3 = e^{-2t}$$

From reminder

$$A_3(t) = e^{-k\left(\frac{3\pi}{L}\right)^2 t}\left(\int_0^t e^{-2w} e^{+k\left(\frac{3\pi}{L}\right)^2 w}\,dw + A_3(0)\right).$$

$$\int_0^t e^{\left(k\left(\frac{3\pi}{L}\right)^2 - 2\right)w}\,dw = \frac{e^{\left(k\left(\frac{3\pi}{L}\right)^2-2\right)w}}{k\left(\frac{3\pi}{L}\right)^2 - 2}\Bigg|_0^t = \frac{e^{\left(k\left(\frac{3\pi}{L}\right)^2-2\right)t}}{k\left(\frac{3\pi}{L}\right)^2 - 2} - 1$$

$$A_3(t) = e^{-k\left(\frac{3\pi}{L}\right)^2 t}\left[\left(\frac{e^{\left(k\left(\frac{3\pi}{L}\right)^2-2\right)t}}{k\left(\frac{3\pi}{L}\right)^2-2} - 1\right) + A_3(0)\right]$$

$$= A_3(0)\, e^{-k\left(\frac{3\pi}{L}\right)^2 t} + \frac{e^{-2t} - e^{-k\left(\frac{3\pi}{L}\right)^2 t}}{k\left(\frac{3\pi}{L}\right)^2 - 2}$$

provided $k\left(\frac{3\pi}{L}\right)^2 \neq 2.$

i.e. e^{-2t} is not a homogeneous solution.

$n \neq 0, 3$

$$\frac{d}{dt} A_n(t) + k\left(\frac{n\pi}{L}\right)^2 A_n = 0$$

$$A_n'(t) = -k\left(\frac{n\pi}{L}\right)^2 A_n(t)$$

$$\frac{A_n'(t)}{A_n(t)} = -k\left(\frac{n\pi}{L}\right)^2$$

$$\frac{d}{dt}\left(\ln(A_n(t))\right) = -k\left(\frac{n\pi}{L}\right)^2$$

$$\boxed{A_n(t) = A_n(0)\, e^{-k\left(\frac{n\pi}{L}\right)^2 t}}$$

for the solution

$$u(x,t) = \sum_{n=0}^{\infty} A_n(t) \cos\left(\frac{n\pi x}{L}\right).$$

Term by term integration

Thm

A fourier series of a piecewise smooth function, $f(x)$ can always be integrated term by term and the result is a convergent infinite series that always converges to $\int_{-L}^{L} f(x)\,dx$

Remarks

1. The new series is continuous.
2. The new series may not be a Fourier series.

Example

$$1 \sim \frac{4}{\pi}\left(\sin\left(\frac{\pi x}{L}\right) + \frac{1}{3}\sin\left(\frac{3\pi x}{L}\right) + \frac{1}{5}\sin\left(\frac{5\pi x}{L}\right) + \cdots\right)$$

Integrating the series term by term from 0 to x

$$x \sim \frac{4}{\pi}\int_0^x \left(\sin\left(\frac{\pi t}{L}\right) + \frac{1}{3}\sin\left(\frac{3\pi t}{L}\right) + \frac{1}{5}\sin\left(\frac{5\pi t}{L}\right) + \cdots\right) dt$$

$$-\frac{4L}{\pi^2}\left(\cos\left(\frac{\pi t}{L}\right) + \frac{1}{3^2}\cos\left(\frac{3\pi t}{L}\right) + \frac{1}{5^2}\sin\left(\frac{5\pi t}{L}\right) + \cdots\right)\Bigg|_0^x$$

$$\frac{4L}{\pi^2}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right) - \frac{4L}{\pi^2}\left(\cos\left(\frac{\pi x}{L}\right) + \frac{1}{3^2}\cos\left(\frac{3\pi x}{L}\right) + \frac{1}{5^2}\cos\left(\frac{5\pi x}{L}\right) + \cdots\right)$$

Notice that
The series on the right hand side should be the Fourier cosine series of x but there is an extra series of constants.

Recall the Fourier cosine series of x

$$x \sim \frac{L}{2} - \frac{4L}{\pi^2}\sum_{n \text{ odd}} \frac{1}{n^2}\cos\left(\frac{n\pi x}{L}\right)$$

aside $\sum_{n \text{ odd}} \frac{1}{n^2}\cos\left(\frac{n\pi x}{L}\right)$ can be written as $\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}\cos\left(\frac{(2k+1)\pi x}{L}\right)$

We can evaluate the infinite series as follows

$$x = \$ - \frac{4L}{\pi^2}\left(\cos\left(\frac{\pi x}{L}\right) + \frac{1}{3^2}\cos\left(\frac{3\pi x}{L}\right) + \frac{1}{5^2}\cos\left(\frac{5\pi x}{L}\right) + \dots\right)$$

$$\int_0^L x\,dx = \int_0^L \$\,dx - \frac{4L}{\pi^2}\int_0^L \left(\cos\left(\frac{\pi x}{L}\right) + \frac{1}{3^2}\cos\left(\frac{3\pi x}{L}\right) + \frac{1}{5^2}\cos\left(\frac{5\pi x}{L}\right) + \dots\right)dx$$

$$\int_0^L S\,dx = \frac{4L}{\pi^2}\left(\sin\left(\frac{\pi x}{L}\right)\Big|_0^L + \frac{1}{3^2}\cos\left(\frac{3\pi x}{L}\right)\Big|_0^L + \frac{1}{5^2}\cos\left(\frac{5\pi x}{L}\right)\Big|_0^L + \dots\right)$$

$$\$L = \frac{4L}{\pi^2}\left(0 + \dots 0\right)$$

so $\$ = \frac{1}{L}\int_0^L x\,dx = \frac{1}{L}\frac{x^2}{2}\Big|_0^L = \frac{L}{2}.$

$= \cancel{L}$

so

$$x = \frac{L}{2} - \frac{4L}{\pi^2}\left(\cos\left(\frac{\pi x}{L}\right) + \frac{1}{3^2}\cos\left(\frac{3\pi x}{L}\right) + \frac{1}{5^2}\cos\left(\frac{5\pi x}{L}\right) + \dots\right)$$

Notice that we can derive a new series as follows:

$$\int_0^x t\,dx = \int_0^x \frac{L}{2}\,dx - \frac{4L}{\pi^2}\int_0^x \left(\cos\left(\frac{\pi t}{L}\right) + \frac{1}{3^2}\cos\left(\frac{3\pi t}{L}\right) + \frac{1}{5^2}\cos\left(\frac{5\pi t}{L}\right) + \dots\right)dt$$

$$\frac{x^2}{2} = \frac{L}{2}x - \frac{4L^2}{\pi^3}\left(\sin\left(\frac{\pi x}{L}\right) + \frac{1}{3^3}\sin\left(\frac{3\pi x}{L}\right) + \frac{1}{5^3}\sin\left(\frac{5\pi x}{L}\right) + \dots\right)$$

Complex form of Fourier series

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$$

where $a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \quad \& \quad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

so that

$$f(x) \sim a_0 + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n + \frac{b_n}{i}\right)e^{i\left(\frac{n\pi x}{L}\right)} + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n - \frac{b_n}{i}\right)e^{-i\left(\frac{n\pi x}{L}\right)}$$

$$\sim a_0 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n - ib_n)\,e^{i\left(\frac{n\pi x}{L}\right)} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n + ib_n)\,e^{-i\left(\frac{n\pi x}{L}\right)}$$

$$\sim a_0 + \frac{1}{2}\sum_{n=-1}^{-\infty}(a_{(-n)} - ib_{(-n)})\,e^{-i\left(\frac{n\pi x}{L}\right)} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n + ib_n)\,e^{-i\left(\frac{n\pi x}{L}\right)}$$

⋮

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{-i\left(\frac{n\pi x}{L}\right)}.$$