5.8 Boundary conditions of the third kind

So far

1. Constant coefficients
   - Dirichet $\longrightarrow$ Fourier sine series
   - Neumann $\longrightarrow$ Fourier cosine series
   - Periodic $\longrightarrow$ Fourier series.

2. Variable coefficients leading to the (SLP) problem.

Back to the Heat Equation

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}.$$

or Wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

with

$$u(0,t) = 0 \qquad (1)$$

$$\frac{\partial u}{\partial x}(L,t) = -h\,u(L,t). \qquad (2)$$

(i) For the heat problem (2) corresponds to Newton's Law of cooling

(ii) Vibrating string - restoring force if $h>0$

For physical problems $h \geq 0$ but mathematically we analyze $h<0$ case.

Eigen value Problem

$$u(x,t) = G(t)\,\phi(x)$$

Heat flow

$$\frac{dG}{dt} = -\lambda k G$$

String

$$\frac{d^2 G}{dt^2} = -\lambda c^2 G.$$

The spatial problem in both cases is

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0$$

$$\phi(0) = 0$$

$$\frac{d\phi}{dx}(L) + h\phi(L) = 0$$

$h \geq 0 \rightsquigarrow$ physical case

$h < 0 \rightsquigarrow$ non-physical case.

$\underline{\lambda > 0}$

$$\phi(x) = c_1 \cos\sqrt{\lambda}\,x + c_2 \sin\sqrt{\lambda}\,x$$

$\phi(0) = 0 \Rightarrow c_1 = 0$ and hence

$$\boxed{\phi(x) = c_2 \sin\sqrt{\lambda}\,x.}$$

we need the sine functions to satisfy $\phi(0) = 0$!

$$\frac{d\phi}{dx} = c_2\sqrt{\lambda}\cos(\sqrt{\lambda}\,x)$$

So the second boundary condition implies

$$c_2\left(\sqrt{\lambda}\cos(\sqrt{\lambda}\,L) + h\sin\sqrt{\lambda}\,L\right) = 0.$$

If $c_2 \equiv 0$ then $\phi \equiv 0$ (not an eigenfunction)

So the eigenfunctions exist and satisfy

$$\sqrt{\lambda}\cos(\sqrt{\lambda}\,L) + h\sin\sqrt{\lambda}\,L = 0. \qquad (*)$$

To solve $(*)$ divide by $\cos(\sqrt{\lambda}\,L)$ (because $\cos(\sqrt{\lambda}\,L) \neq 0$).

$$\sqrt{\lambda} + h\tan\sqrt{\lambda}\,L = 0$$

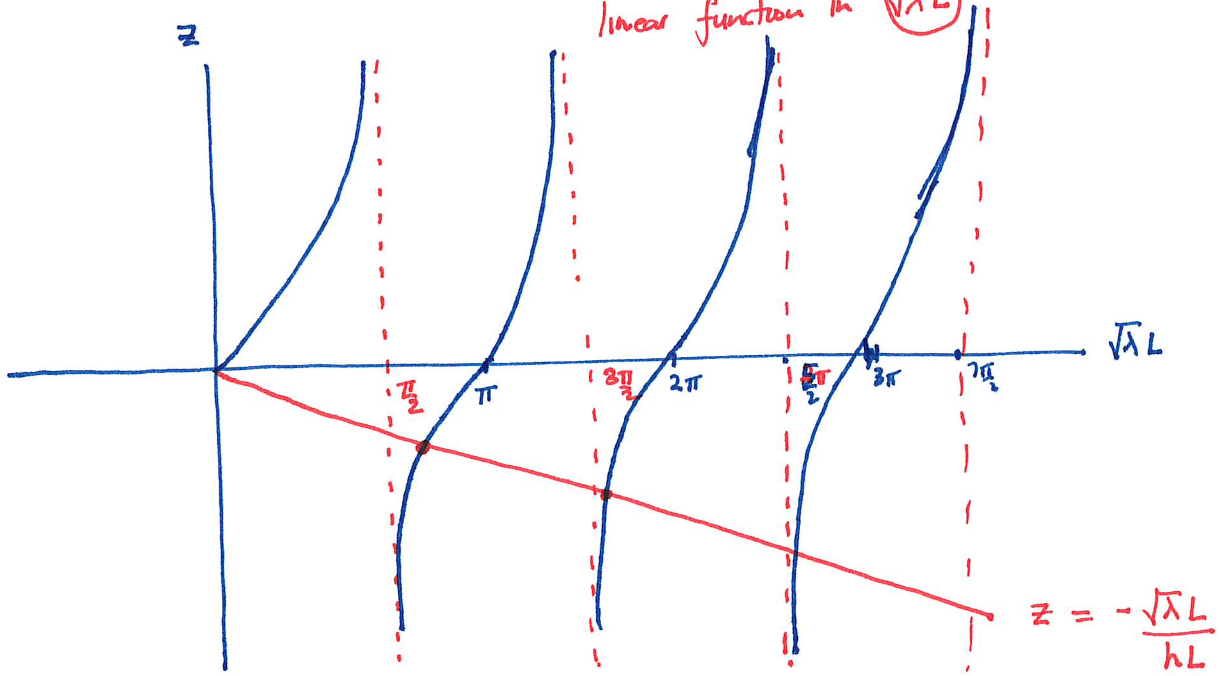
$$\Rightarrow \tan\sqrt{\lambda}\,L = -\frac{\sqrt{\lambda}}{h}$$

Solving $\qquad \tan\sqrt{\lambda}L = -\dfrac{\sqrt{\lambda}}{h}$

Let $\quad z = \tan(\sqrt{\lambda}L) \qquad$ and $\qquad z = -\dfrac{\sqrt{\lambda}}{h} \cdot \dfrac{L}{L}$

$$z = -\frac{\sqrt{\lambda}L}{hL}$$

linear function in $\sqrt{\lambda}L$



From the graph

$$\frac{\pi}{2} < \sqrt{\lambda_1}L < \pi$$

$$\frac{3\pi}{2} < \sqrt{\lambda_2}L < 2\pi$$

Notice that as $n \to \infty \qquad \sqrt{\lambda_n}L \sim \left(n - \frac{1}{2}\right)\pi$

In practice we can solve using Newton for small n and use the formula

$$\sqrt{\lambda_n}L = \left(n - \frac{1}{2}\right)\pi$$

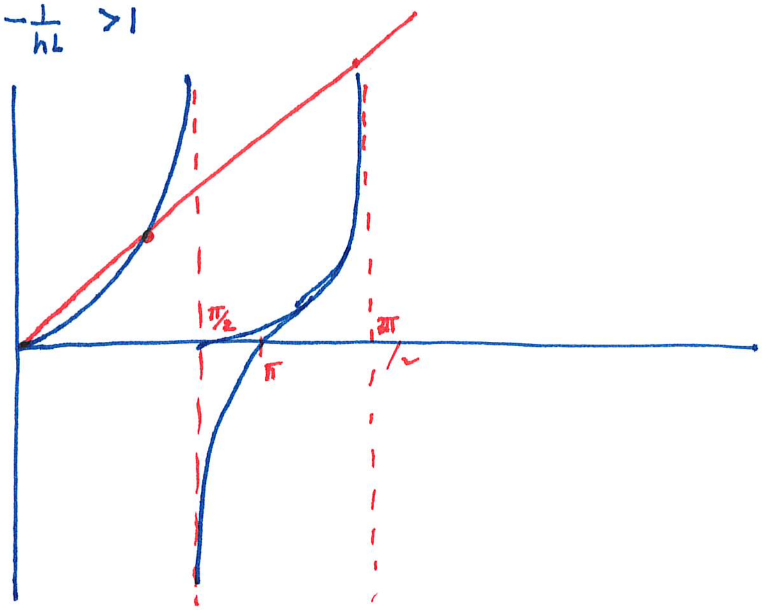
for reasonably large n.

$h<0$

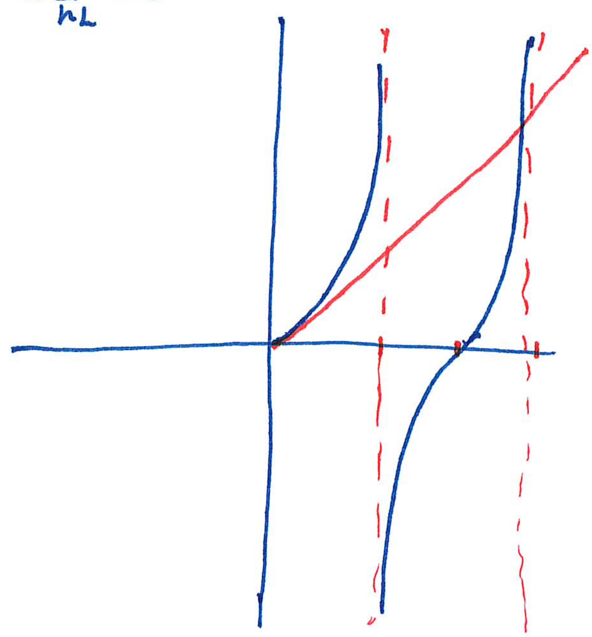
$z = \tan\sqrt{\lambda} L \qquad z = -\dfrac{\sqrt{\lambda} L}{hL}$

Case 1

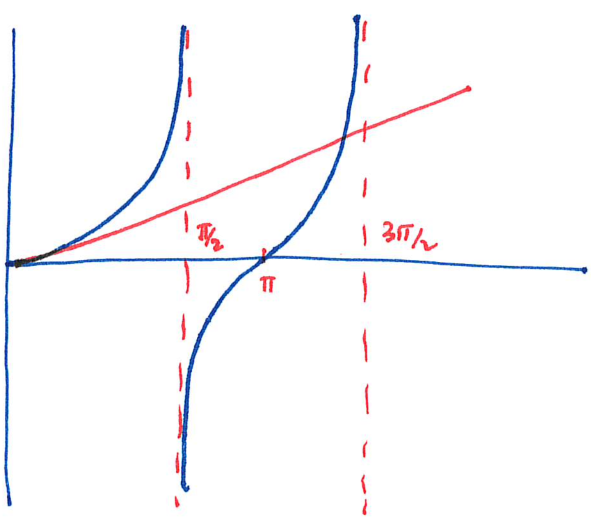
$-\dfrac{1}{hL} > 1$



$-\dfrac{1}{hL} = 0$



$-\dfrac{1}{hL} < 0$



As before $\quad \sqrt{\lambda_n}\, L \sim \left(n - \frac{1}{2}\right)\pi \quad$ as $n \to \infty$

We have an infinite number of eigenvalues with $\phi_n(x) = \sin\sqrt{\lambda}\, x$.

Example : Wave Equation

$h > 0$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial u}{\partial x}(0,t) - h\, u(0,t) = 0$$

$$\frac{\partial u}{\partial x}(L,t) = 0 \qquad \frac{\partial u}{\partial t}(x,0) = g(x).$$

(a) Show that there are an infinite number of different frequencies of oscillation.

(b) Estimate the large frequencies of oscillation

(c) Solve the initial value problem.

From separation of variables, $u(x,t) = \phi(x)\, G(t)$

$$\frac{1}{c^2 G}\frac{d^2 G}{dt^2} = \frac{1}{\phi}\frac{d^2\phi}{dx^2} = -\lambda.$$

$$\frac{1}{\phi}\frac{d^2\phi}{dx^2} = -\lambda \quad \Rightarrow \quad \frac{d^2\phi}{dx^2} + \lambda\phi = 0$$

$$\frac{d\phi}{dx}(0) - h\phi(0) = 0 \qquad (1)$$

$$\frac{d\phi}{dx}(L) = 0. \qquad (2)$$

Since $\lambda > 0$,

$$\phi(x) = c_1 \cos\sqrt{\lambda}\, x + c_2 \sin\sqrt{\lambda}\, x.$$

$$\frac{d\phi}{dx} = -c_1\sqrt{\lambda}\sin\sqrt{\lambda}\, x + c_2\sqrt{\lambda}\cos\sqrt{\lambda}\, x$$

From the first B.C

$$\frac{d\phi}{dx}(0) - h\phi(0) \Rightarrow c_2\sqrt{\lambda} - hc_1 = 0 \Rightarrow \boxed{c_1 = \frac{\sqrt{\lambda}}{h} c_2} \Rightarrow \frac{c_2}{c_1} = \frac{h}{\sqrt{\lambda}}$$

The second B.C implies

$$-c_1\sqrt{\lambda}\sin\sqrt{\lambda}L + c_2\sqrt{\lambda}\cos\sqrt{\lambda}L = 0.$$

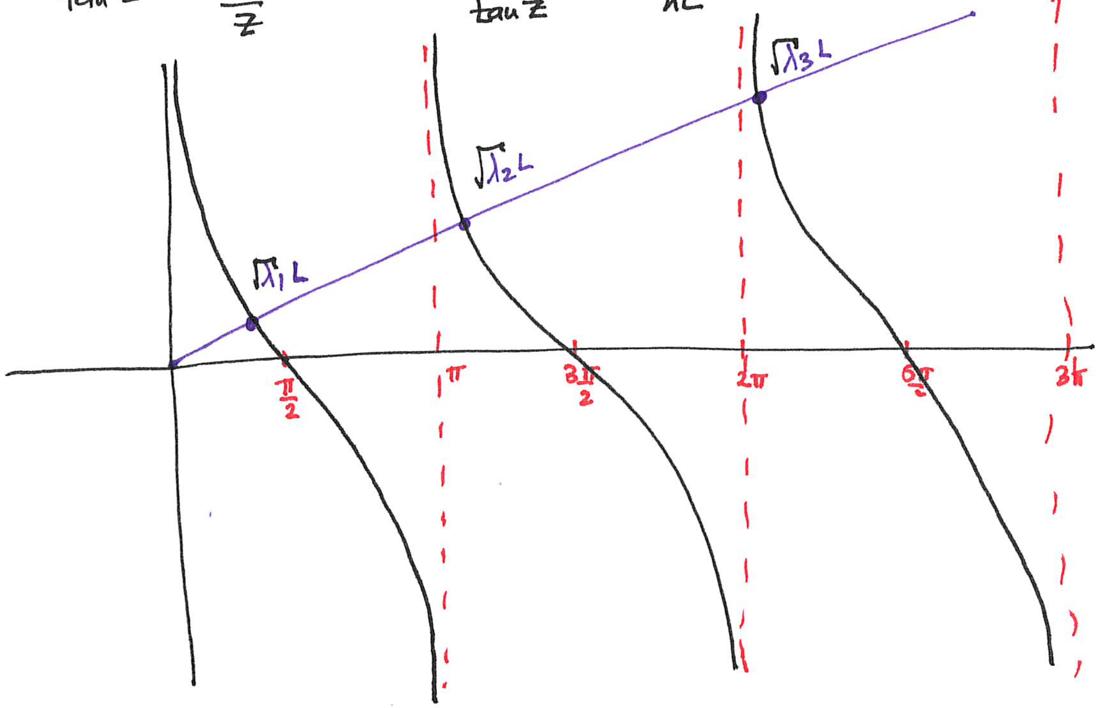
$$C_2\sqrt{\lambda}\cos(\sqrt{\lambda}L) = C_1\sqrt{\lambda}\sin(\sqrt{\lambda}L)$$

$$\tan\sqrt{\lambda}L = \frac{C_2}{C_1} = \frac{h}{\sqrt{\lambda}}$$

$$\tan\sqrt{\lambda}L = \frac{h}{\sqrt{\lambda}} \Rightarrow \tan\sqrt{\lambda}L = \frac{hL}{\sqrt{\lambda}L}$$

Let $z = \sqrt{\lambda}L$

$$\tan z = \frac{hL}{z} \Rightarrow \frac{1}{\tan z} = \frac{z}{hL} \Rightarrow \cot z = \frac{z}{hL}.$$



For large n $\qquad$ for large n

$$\sqrt{\lambda_n}L \sim (n-1)\pi \quad \text{so} \quad \lambda_n \simeq \left(\frac{(n-1)\pi}{L}\right)^2, \quad n = 2, 3, \ldots$$

hich proves that there are an infinite number of frequencies of illation.

ssuming we have the eigen values, then

$$\phi(x) = C_1\cos\sqrt{\lambda_n}x + \frac{\sqrt{\lambda}}{h}C_1\sin(\sqrt{\lambda_n})x$$

$$= C_1\left(\cos\sqrt{\lambda_n}x + \frac{\sqrt{\lambda}}{h}\sin\sqrt{\lambda_n}x\right)$$

The time dependent problem is of the form

$$\frac{1}{c^2 G} \cdot \frac{d^2 G}{dt^2} = -\lambda, \qquad \lambda > 0$$

has a general solution

$$G(t) = d_1 \cos(\sqrt{\lambda_n}\, ct) + d_2 \sin(\sqrt{\lambda_n}\, ct)$$

So our product solution takes the form

$$u(x,t) = \phi(x) G(t)$$

then super-position means

$$u(x,t) = \sum_{n=1}^{\infty} \phi_n(x) \left(a_n \cos\sqrt{\lambda_n}\, ct + b_n \sin(\sqrt{\lambda}\, ct) \right)$$

From the initial conditions :

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} \phi_n(x)\, a_n$$

$$\int_0^L f(x)\phi_m(x)\,dx = \sum_{n=1}^{\infty} a_n \int_0^L \phi_n(x)\,\phi_m(x)\,dx$$

$$\int_0^L f(x)\,\phi_n(x)\,dx = a_n \int_0^L \phi_n^2\,dx$$ because $\{\phi_n\}$ is orthogonal.

$$a_n = \frac{\int_0^L f(x)\,\phi_n(x)\,dx}{\int_0^L \phi_n(x)^2\,dx}$$

Similarly

$$g(x) = \frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} \phi_n(x)\, b_n \sqrt{\lambda_n}\, c \cos(\sqrt{\lambda}\, ct)$$

$$\vdots$$

$$\frac{\partial u}{\partial t}(x,t) = \sum_{n=1}^{\infty} \phi_n(x) \cdot a_n \sqrt{\lambda_n}\, c(-1)\sin(\sqrt{\lambda_n}\, ct) + b_n \sqrt{\lambda}\, c \cos(\sqrt{\lambda}\, ct)$$

$$b_n \sqrt{\lambda_n}\, c = \frac{\int_0^L g(x)\,\phi_n(x)\,dx}{\int_0^L \phi_n(x)^2\,dx}$$