Non-homogeneous Problems

Chapter 8.

Most physical problems are time dependent and have non-homogeneous boundary conditions.

Example #1

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

$$u(0,t) = A \quad , \quad u(L,t) = B$$

$$u(x,0) = f(x).$$

Equilibrium Temperature

Let $u_E(x)$ be the equilibrium temperature distribution.

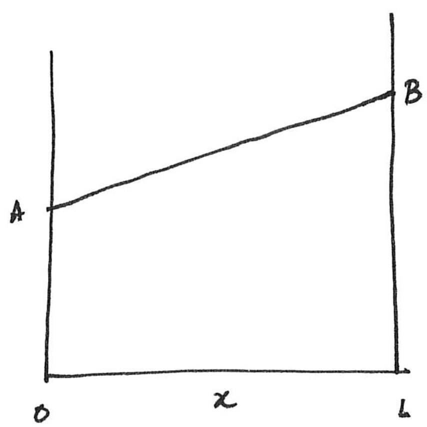
$$\frac{d^2 u_E}{dx^2} = 0 \quad \text{because} \quad \frac{\partial u}{\partial t} = 0.$$

$$u_E(0) = A$$

$$u_E(L) = B$$

WLOG, suppose $A < B$

$$u_E(x) = A + \frac{B-A}{L} x$$



Displacement from Equilibrium

$$v(x,t) = u(x,t) - u_E(x,t)$$

We solve for $v(x,t)$, since $u_E(x,t)$ is linear in x,

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} \Rightarrow \boxed{\frac{\partial v}{\partial t} = k \frac{\partial^2 v}{\partial x^2}}$$

$$v(x,t) = u(x,t) - u_E(x,t)$$

$$v(0,t) = u(0,t) - u_E(0) = 0 \quad , \text{ similarly}$$

$$v(L,t) = 0.$$

@ $t=0$, $v(x,t)$ is the difference between the equilibrium temperature and the initial temperature.

$$v(x,0) = f(x) - u_E(x).$$

Now the problem

$$\boxed{\begin{aligned} \frac{\partial v}{\partial t} &= k\frac{\partial^2 v}{\partial x^2} \\ v(0,t) &= 0 \\ v(L,t) &= 0 \\ v(x,0) &= f(x) - u_E(x) \end{aligned}}$$

is linear an homogeneous

$$v(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

where the initial conditions satisfy

$$f(x) - u_E(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)$$

so that

$$a_n = \frac{2}{L}\int_0^L [f(x) - u_E(x)]\sin\left(\frac{n\pi x}{L}\right)dx$$

so

$$\boxed{u(x,t) = u_E(x) + \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}}$$

* The temperature approaches its equilibrium distribution for all initial conditions

i.e $\lim_{t\to\infty} u(x,t) = u_E(x).$

Non Steady non-homogeneous terms

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x,t)$$

$$u(0,t) = A(t)$$
$$u(L,t) = B(t)$$
$$u(x,0) = f(x)$$

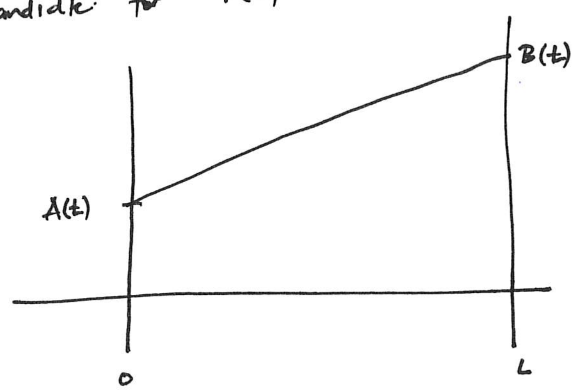
Claim

- We cannot always reduce the problem to a homogeneous PDE with homogeneous boundary conditions.
- We can transform the boundary conditions to homogeneous.

Let $r(x,t)$ be a reference temperature distribution satisfying

$$r(0,t) = A(t)$$
$$r(L,t) = B(t)$$

a possible candidate for $r(t)$



$$r(x,t) = A(t) + \frac{x}{L}\left[B(t) - A(t)\right]$$

$r(x,t)$ is not necessarily the equilibrium temperature distribution.

Let $v(x,t) \equiv u(x,t) - r(x,t)$ [the difference between the chosen solution and the desired solution]

$$v(0,t) = 0$$
$$v(L,t) = 0$$

$$u(x,t) = v(x,t) + r(x,t)$$

$$\frac{\partial v}{\partial t} + \frac{\partial r}{\partial t} = k\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 r}{\partial x^2}\right] + Q(x,t)$$

So

$$\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2} + \left[Q(x,t) - \frac{\partial r}{\partial t} + k\frac{\partial^2 r}{\partial x^2}\right] = k\frac{\partial^2 v}{\partial x^2} + Q^*$$

$$v(x,0) = f(x) - r(x,0) = f(x) - A(0) - \frac{x}{L}\left[B(0) - A(0)\right] \equiv g(x).$$

Method of eigenfunction expansion

The relatedted homogeneous problem

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

$$u(0,t) = 0$$
$$u(L,t) = 0$$

has eigenfunctions satisfying

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0$$
$$\phi(0) = \phi(L) = 0.$$

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1,2,3,\ldots \quad \text{with} \quad \phi_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

However the eigenfunctions for ~~our~~ other problem may be different, but from our theory of Sturm-Liouville problems we know that $\phi_n(x)$ exists.

Let (*) $v(x,t) = \sum_{n=1}^{\infty} a_n(t)\,\phi_n(x)$ (in our case $\phi_n(x) = \sin\left(\frac{n\pi x}{L}\right)$)

Note

(*) may be similar to the product solutions from separation of variables but this is not the case here!

The initial conditions are satisfied if

$$g(x) = \sum_{n=1}^{\infty} a_n(0)\,\phi_n(x)$$

where

$$a_n(0) = \frac{\int_0^L g(x)\,\phi_n(x)\,dx}{\int_0^L \phi_n^2(x)\,dx}.$$

We will determine $a_n(t)$ by direct substitution (which requires term-by term differentiation).

~~FACT If $v(t)$ solves the same homogeneous boundary~~

$$v(x,t) = \sum_{n=1}^{\infty} a_n(t)\,\phi_n(x), \quad v(0) = v(L) = 0$$

So we can differentiate term by term

$$\frac{\partial v}{\partial t} = \sum_{n=1}^{\infty} \frac{d}{dt} a_n(t)\, \phi_n(x) \qquad \left(\text{assuming } a_n'(t) \text{ is piecewise smooth}\right)$$

$$\frac{\partial^2 v}{\partial x^2} = \sum_{n=1}^{\infty} a_n(t) \frac{d^2\phi_n(x)}{dx^2} = -\sum_{n=1}^{\infty} a_n(t)\, \lambda_n\, \phi_n(x)$$

$$\uparrow$$

$$\frac{d^2\phi_n}{dx^2} + \lambda_n \phi_n = 0$$

FACT: If v, $\frac{\partial v}{\partial x}$ are continuous and if $v(x,t)$ solves the same homogeneous b.c (as $\phi_n(x)$) then the necessary term by term diff is justified

then plugging into the non-homogeneous problem

$$\sum_{n=1}^{\infty} \left[\frac{da_n}{dt} + \lambda_n k\, a_n \right] \phi_n(x) = Q^*(x,t)$$

The left-hand side is a generalized Fourier series of $Q^*(x,t)$. so

$$\frac{da_n}{dt} + \lambda_n k a_n = \frac{\int_0^L Q^*(x,t)\, \phi_n(x)\, dx}{\int_0^L \phi_n^2(x)\, dx} = \bar{q}_n(t)$$

We need to solve

$$\frac{da_n}{dt} + \lambda_n k a_n = \bar{q}_n$$

$$e^{\lambda_n k t}\left(\frac{da_n}{dt} + \lambda_n k a_n \right) = \frac{d}{dt}\left(a_n e^{\lambda_n k t} \right) = \bar{q}_n e^{\lambda_n k t}$$

So $$a_n(t)\, e^{\lambda_n k t} - a_n(0) = \int_0^t \bar{q}_n(\tau)\, e^{\lambda_n k \tau}\, d\tau$$

So $$a_n(t) = a_n(0)\, e^{-\lambda_n k t} + e^{-\lambda_n k t} \int_0^t \bar{q}_n(\tau)\, e^{\lambda_n k \tau}\, d\tau$$

So $$v(x,t) = \sum_{n=1}^{\infty} a_n(t)\, \phi_n(x).$$

If the problem is homogeneous, $Q(x,t) = 0$, then

$$v(x,t) = \sum_{n=1}^{\infty} a_n(t)\, \phi_n(x) \quad \text{where} \quad a_n(t) = a_n(0)\, e^{-\lambda_n k t}$$

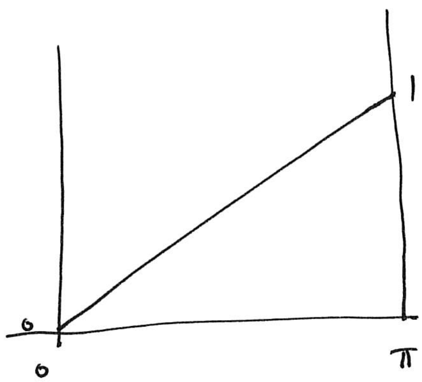
Example

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + e^{-t}\sin(3x)$$

subject to

$$u(0,t) = 0$$
$$u(\pi,t) = 1$$
$$u(x,0) = f(x)$$

To convert the problem to a homogeneous problem



$r(x) = \frac{x}{\pi}$

we introduce a reference solution $r(x) = \frac{x}{\pi}$. and let

$v(x) = u(x,t) - \frac{x}{\pi}$ so that

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + e^{-t}\sin(3x)$$

subject to

$$v(0,t) = 0$$
$$v(1,t) = 0$$
$$u(x,0) = f(x) - \frac{x}{\pi}.$$

In this case $\phi_n(x) = \sin\left(\frac{n\pi x}{L}\right) = \sin(nx)$ $(L=\pi)$ so we expand the solution

$$v(x,t) = \sum_{n=1}^{\infty} a_n(t)\,\phi_n(t) = \sum_{n=1}^{\infty} a_n(t)\sin nx.$$

After term by term differentiation and plugging into the PDE

$$\sum_{n=1}^{\infty} \left(\frac{da_n}{dt} + n^2 a_n \right) \sin nx = e^{-t} \sin(3x)$$

so

$$\frac{da_n}{dt} + n^2 a_n = \begin{cases} 0, & n \neq 3 \\ e^{-t}, & n=3. \end{cases}$$

① $a_n'(t) + n^2 a_n(t) = 0.$

$$\frac{a_n'(t)}{a_n(t)} = -n^2$$

$$\frac{d}{dt}\left(\ln(a_n(t))\right) = -n^2 \quad \Rightarrow \boxed{a_n(t) = a_n(0)\, e^{-n^2 t}} \quad n \neq 3.$$

/

② $a_n'(t) + n^2 a_n = e^{-t}$. (Integrating factor)

$$a_3'(t) + 9a_n = e^{-t}$$

$$a_3(t) = \frac{1}{8} e^{-t} + \left[a_3(0) - \frac{1}{8} \right] e^{-9t}.$$

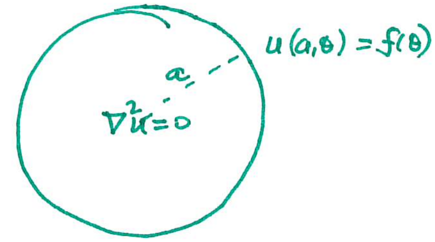
Where $a_n(0) = \frac{2}{\pi} \int_0^{\pi} \left[f(x) - \frac{x}{\pi} \right] \sin(nx)\, dx.$

so $u(x,t) = v(x,t) + \frac{x}{\pi}$

$$\Rightarrow \boxed{u(x,t) = \sum_{n=1}^{\infty} a_n(t)\, \phi_n(x) + \frac{x}{\pi}}$$

What if the method of separation of variables fails?

Back to Laplace on a disk.



$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

$$u(r,-\pi) = u(r,\pi)$$

$$\frac{\partial u}{\partial \theta}(r,-\pi) = \frac{\partial u}{\partial \theta}(r,\pi)$$

$$|u(0,\theta)| < \infty, \quad u(a,\theta) = f(\theta).$$

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} A_n r^n \cos n\theta + \sum_{n=1}^{\infty} B_n r^n \sin n\theta \qquad 0 \le r < a, \quad -\pi < \theta \le \pi.$$

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta$$

$$A_n a^n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\,d\theta$$

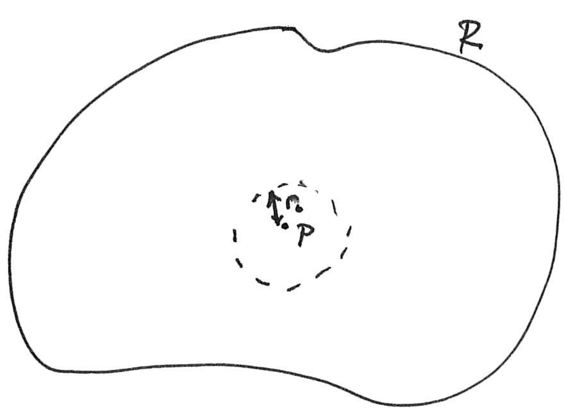
$$B_n a^n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)\,d\theta$$

$$\boxed{u(0,\theta) = A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta} \; *$$

The temperature at the middle of the circle = the average of the temperature at the edges of the circle.

This is called the Mean Value Property of Laplace's equation.

This property holds in general throughout the region R



For any point $P \in R$, the temperature at P is the average of the temperature along any circle of radius r_0

Maximum principles

1. In the steady state (assuming no sources), the temperature cannot attain its maximum in the interior (unless it is a constant).

PROOF

Suppose the maximum was at the point P, however this should be the average of all points on the circle. It is impossible for the temperature at P to be larger. This contradicts the original assumption.

One can also show that the minimum temperature cannot occur in the interior.

FACT

In the steady state, the maximum and minimum temperatures occur on the boundary.

Well-posedness and Uniqueness

A problem is well-posed if there exists a unique solution that depends continuously on the non-homogeneous data.

We can use the Maximum principle to show that

$$\nabla^2 u = 0 \quad \text{in } \Omega$$
$$u = f(x), \quad \text{on } \partial\Omega.$$

is well-posed.

Suppose we vary the boundary by a small amount so that

$$\nabla^2 v = 0 \quad \text{in } \Omega$$
$$u = g(x) \quad \text{on } \partial\Omega$$

Let $w = v - u$, due to linearity
$\nabla^2 w = 0$
$w = f(x) - g(x)$.

The maximum (and minimum) principles for Laplace's equation imply that the maximum and minimum occur at the boundary so

$$\min\left(f(x) - g(x)\right) \leq w \leq \max\left(f(x) - g(x)\right)$$

Since $g(x)$ is close to $f(x)$, w is small so the solution u is close to v.

Laplace's equation is unique

Suppose there are 2 solutions u and v satisfying Laplace's equation

i.e

$$\nabla^2 v = 0 \qquad \nabla^2 w = 0$$
$$v = g(x) \qquad w = f(x)$$

by linearity we have $\nabla^2 w = 0$
$w = 0$ on $\partial\Omega$.

By the max/min principle $0 \leq w \leq 0 \Rightarrow w = 0$ everywhere inside.

This implies $u = v$, proving that if a solution exists, it must be unique.

Solvability condition

If on the boundary the heat flow

$-K_0 \nabla u \cdot n$ is specified

Laplace's equation may have no solutions.

$$\iint \nabla^2 u = 0 \, dx\,dy = \iint \nabla \cdot (\nabla u) \, dx\,dy$$

Using the divergence theorem

$$0 = \oint \nabla u \cdot \hat{n} \, ds$$

However $\nabla u \cdot n$ is proportional to heat flow through the boundary so for the steady state solution to exist, the net heat flow through the boundary must be zero.

This is called the compatibility condition.