# 1.4 Equilibrium Temperature Distribution

- constant thermal coefficients and no sources/sinks.

Consider

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

$$u(x,0) = f(x) \quad \text{(initial condition)} \qquad (P)$$

$$\boxed{\begin{aligned} u(0,t) &= T_1(t) \\ u(L,t) &= T_2(t) \end{aligned}} \quad \text{Boundary conditions}$$

$$k = \frac{K_0}{\rho c}$$

The eventual goal is to solve P, for now we consider a simpler problem

~~Goal~~ Find the equilibrium temperature distribution if the boundary conditions are steady (independent of temperature)

i.e

$$u(0,t) = T_1 \quad \text{and} \quad u(L,t) = T_2.$$

## Steady state/ equilibrium solution

The temperature distribution that is independent of time, $t$.

i.e $\quad u(x,t) = u(x) \Rightarrow \frac{\partial u}{\partial t} = 0$

so that (P) becomes an ODE!

$$\frac{d^2u}{dx^2} = 0 \quad \text{with boundary conditions} \quad (P_s)$$

$$\boxed{u(0) = T_1 \quad \text{and} \quad u(L) = T_2}$$

In solving $P_s$, we usually ignore the initial condition.

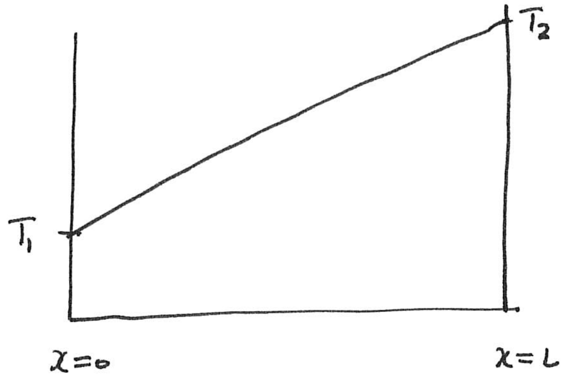
$$\frac{d^2u}{dx^2} = 0 \Rightarrow u(x) = C_1 x + C_2$$

Applying boundary conditions

$$u(0) = T_1 \Rightarrow C_2 = T_1$$

$$u(L) = T_2 \Rightarrow C_1 L + T_1 = T_2 \Rightarrow C_1 = \frac{T_2 - T_1}{L}$$

So the solution to $P_s$ is $\quad u(x) = T_1 + \frac{T_2 - T_1}{L} x$




## Remark

For the time dependent problem with steady boundary conditions

$$\boxed{\begin{aligned} \frac{\partial u}{\partial t} &= k \frac{\partial^2 u}{\partial x^2} \\ u(x,0) &= f(x) \\ u(0,t) &= T_1 , \quad u(L,t) = T_2 \end{aligned}}$$

We expect

$$\lim_{t \to \infty} u(x,t) = u(x) = T_1 + \left(\frac{T_2 - T_1}{L}\right) x$$

i.e in the long run the $u(x,t)$ does not remain equal to the initial distribution.
The steady state boundary conditions dominate.

## Insulated Boundaries

1D rod with no sources insulated at the boundaries

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

$$u(x,0) = f(x)$$

B.C

$$\frac{\partial u}{\partial x}(0,t) = 0$$

$$\frac{\partial u}{\partial x}(L,t) = 0$$

Find the Equilibrium temperature

1. Set $\frac{\partial u}{\partial t} = 0$ so that

$$(P_x) \qquad \frac{d^2u}{dx^2} = 0 \qquad (\text{ODE})$$

Boundary conditions: $\frac{du}{dx}(0) = 0$ & $\frac{du}{dx}(L) = 0$ (neglect initial conditions for the moment)

The general solution of $(P_x)$ is $u(x) = C_1 x + C_2$.

Applying $\frac{du}{dx}(0) = 0 \Rightarrow C_1 = 0$ so $\boxed{u(x) = C_2}$, (we do not have a unique solution)

For the time dependent problem

$$\lim_{t\to\infty} u(x,t) = C_2.$$

REMARK

It makes physical sense that an insulated rod should approach a constant temp as $t \to \infty$. BUT any constant?

To determine the constant we have to look at the initial condition.

If both ends of the rod are insulated, the total thermal energy should remain constant.

start from

$$c\rho \frac{\partial u}{\partial t} = K_0 \frac{\partial^2 u}{\partial x^2} + Q.$$

$$\int_0^L c\rho \frac{\partial u}{\partial t}\, dx = \int_0^L K_0 \frac{\partial^2 u}{\partial x^2}\, dx$$

~~and recall that $k = \frac{K_0}{c\rho}$~~

$$\frac{d}{dt}\int_0^L c\rho u\, dx = K_0 \frac{\partial u}{\partial x}(x,t)\Big|_0^L = -K_0 \frac{\partial u}{\partial x}(0,t) + K_0 \frac{\partial u}{\partial x}(L,t)$$

$$= 0$$

So $\frac{d}{dt}\int_0^L c\rho u\, dx = \text{CONSTANT}.$

$\int_0^L A\, e(x,t)\, dt$

Recall that $e(x,t) = c\rho u(x,t)$ so the total thermal energy should be a constant. So

$$\text{Initial thermal energy} = \text{Final thermal energy}$$

so $$A c \rho \int_0^L u(x,0)\,dx = A c \rho \int_0^L f(x)\,dx \quad \text{(Initial thermal energy)}$$

Final thermal energy

$$A c \rho \int_0^L C_2\,dx = A c \rho \int_0^L f(x)\,dx$$

$$A c \rho C_2 L = A c \rho \int_0^L f(x)\,dx$$

$$\boxed{C_2 = \frac{1}{L}\int_0^L f(x)\,dx}$$

so the steady state solution is

ie the average of the initial temperature distribution.

Problem with a source term

Determine the equilibrium temperature distribution for a uniform one-dimensional rod with constant thermal properties with

$\frac{Q}{K_0} = 1$, $u(0) = T_1$, $u(L) = T_2$, $u(x,0) = f(x)$

$$c\rho \frac{du}{dt} = K_0 \frac{\partial^2 u}{\partial x^2} + Q. \quad \left(\text{set } \frac{du}{dt} = 0\right)$$

$$0 = K_0 \frac{d^2 u}{dx^2} + Q. \qquad [Q = K_0]$$

$$K_0 \frac{d^2 u}{dx^2} = -Q = K_0 \Rightarrow \frac{d^2 u}{dx^2} = -1.$$

Solving:
$$\frac{du}{dx} = -x + C_1, \qquad u(x) = -\frac{x^2}{2} + C_1 x + C_2.$$

Applying boundary conditions yeilds:

$u(0) = T_1 \Rightarrow C_2 = T_1$

$u(L) = T_2 \Rightarrow -\frac{L^2}{2} + C_1 L + C_2 = T_2 \Rightarrow C_1 = \frac{T_2 - T_1 + \frac{L^2}{2}}{L}$

so

$$u(x) = -\frac{x^2}{2} + \left(\frac{T_2 - T_1 + \frac{L^2}{2}}{L}\right) x + T_1.$$

desmos plot.

Example

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 1 , \quad u(x,0) = f(x) , \quad \frac{\partial u}{\partial x}(0,t) = 1 , \quad \frac{\partial u}{\partial x}(L,t) = \beta$$

Determine the equilibrium temperature distribution. For what values of $\beta$ are there solutions.

1. Find the equilibrium temperature dist $\left(\frac{\partial u}{\partial t} = 0\right)$

$$u_{xx} = -1 \Rightarrow \boxed{u_x = -x + c_1} \Rightarrow u(x) = -\frac{x^2}{2} + c_1 x + c_2$$

2. Applying boundary conditions $u_x(0,t) = 1$ $\quad u_x(L) = 0$

$$u_x(0) = 1 \Rightarrow c_1 = 1$$

$u_x(L) = \beta \Rightarrow -L+1 = \beta$ implies that $\beta = -L+1$.

| | | | | |
|---|---|---|---|---|
| rate of change of heat energy | = | heat energy flowing through the boundary per unit time | + | heat energy generated inside per unit time. |

In the case of the steady solution, $\frac{d}{dt}\left[e(x,t) A \Delta x\right] = 0$ so

it must be the case that

$$0 = A(1-\beta) + A\int_0^L 1 \, dx \Rightarrow 1 - \beta = L$$

$$\Rightarrow \underline{\beta = 1 - L}$$

Verify

or integrate the PDE over the rod

$$\frac{d}{dt}\int_0^L u(x,t)\,dx = \int_0^L u_{xx}\,dx + \int_0^L 1\,dx$$

$$= u_x(L) - u_x(0) + L$$

$= \beta - 1 + L$, therefore if $\beta = 1 - L$, the time derivative is zero!

$e(x,t) = c(x)\rho(x)\,u(x,t)$

$\text{Energy} = \int_0^L e(x)\,A\,dx = A\int_0^L c(x)\,\rho(x)\,u(x)\,dx$

At the initial time $A\int_0^L f(x)\,dx$ assuming $c, \rho = 1$

and at equilibrium $A\int_0^L u(x)\,dx$.

These must be the same so

$$A\int_0^L f(x)\,dx = A\int_0^L u(x)\,dx \implies \int_0^L f(x)\,dx = \int_0^L u(x)\,dx.$$

$$\int_0^L f(x)\,dx = \int_0^L \left(-\frac{x^2}{2} + x + c_2\right) dx = -\frac{x^3}{6} + \frac{x^2}{2} + c_2 x \Big|_0^L = -\frac{L^3}{6} + \frac{L^2}{2} + c_2 L$$

$$\int_0^L f(x)\,dx = -\frac{L^3}{6} + \frac{L^2}{2} + c_2 L$$

$$\boxed{\frac{1}{L}\int_0^L f(x)\,dx + \frac{L^2}{6} - \frac{L}{2} = c_2.}$$

The solution is

$$\boxed{u(x) = -\frac{x^2}{2} + x + \frac{1}{L}\int_0^L f(x)\,dx + \frac{L^2}{6} - \frac{L}{2}}$$