HEAT EQUATION IN 2 or 3 DIMENSIONS

$$c\rho \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0 \frac{\partial u}{\partial x}\right) + Q$$

Conservation of heat energy

$$\boxed{\text{Rate of change of heat energy} = \text{heat energy flowing across the boundaries per unit time} + \text{heat energy generated inside per unit time.}}$$

1. Heat Energy

$$\iiint_R e(x,y,z)\,dV = \iiint_R c\rho u(x,y,z)\,dV$$

2. Heat flux

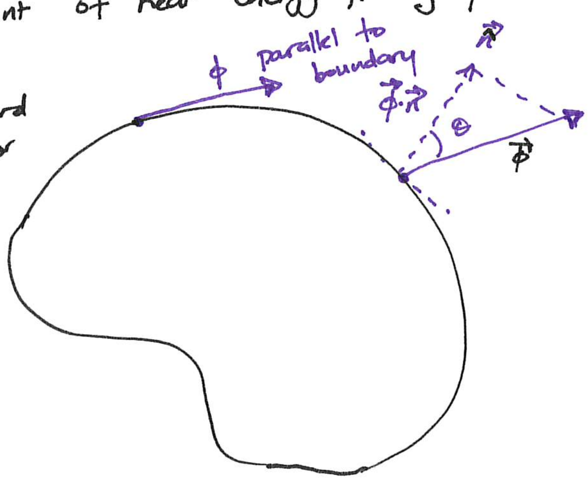
In 1D, $\phi > 0 \Rightarrow$ ~~left~~ right heat flow

$\phi < 0 \Rightarrow$ left heat flow

In 3D, heat flows in some direction, $\vec{v}$, heat flux is a vector, $\vec{\phi}$

$|\vec{\phi}|$ = amount of heat energy flowing per unit time per unit surface area

let $\hat{n}$ – unit outward normal vector



$$\cos\theta = \frac{\vec{n}\cdot\vec{\phi}}{|\vec{n}||\vec{\phi}|}$$

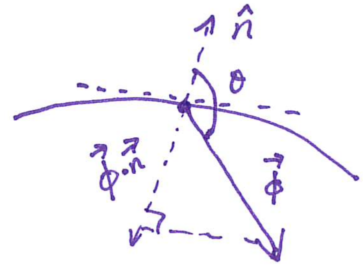
, $\hat{n}$ is the unit normal vector

$$|\vec{\phi}|\cos\theta = \hat{n}\cdot\vec{\phi}$$

The outward normal component of the heat flux vector is

$|\vec{\phi}|\cos\theta = \vec{\phi}\cdot\hat{n}$, $\hat{n}$ is the unit normal vector.

$\vec{\phi}\cdot\hat{n} < 0 \Rightarrow$ outward flow of heat energy is negative $\vec{\phi}$ is directed inwards



$\theta > \frac{\pi}{2} \Rightarrow \cos\theta < 0$ so

$\vec{\phi}\cdot\vec{n} = |\vec{\phi}|\cos\theta < 0.$

Total energy flowing out of a region R per unit time is

$$\oiint \vec{\phi}\cdot\hat{n}\, dS$$

3. <u>HEAT from Sources</u>

$$\iiint_R Q\, dV$$

From the conservation of heat energy

$$\frac{d}{dt}\iiint_R c\rho u\, dV = -\oiint \vec{\phi}\cdot\vec{n}\, dS + \iiint_R Q\, dV$$

<u>Divergence Theorem</u>

In 1D we used FTC to write

$$\phi(a) - \phi(b) = -\int_a^b \frac{\partial\phi}{\partial x}\, dx$$

We want to do the same for

$$\oiint \vec{\phi}\cdot\vec{n}\, dS$$

Recall the divergence operator

Let $\vec{F} = \langle F_1, F_2, F_3\rangle$.

$$\nabla\cdot\vec{F} = \frac{\partial}{\partial x}F_1 + \frac{\partial}{\partial y}F_2 + \frac{\partial}{\partial y}F_3$$

Divergence Theorem (Gauss's theorem)

Volume integral of the divergence of any continuously differentiable vector $\vec{F}$ = Closed surface integral of the outward normal component.

$$\iiint_R \nabla\cdot\vec{F}\,dV = \oint\!\!\oint \vec{F}\cdot\vec{n}\,dS$$

So that

$$\frac{d}{dt}\iiint_R c\rho u\,dV = -\iiint_R \nabla\cdot\vec{\phi}\,dV + \iiint_R Q\,dV$$

$$\iiint_R \left[c\rho\frac{\partial u}{\partial t} + \nabla\cdot\vec{\phi} - Q\right]dV = 0 \Rightarrow c\rho\frac{\partial u}{\partial t} + \nabla\cdot\vec{\phi} - Q = 0$$

$$\boxed{c\rho\frac{\partial u}{\partial t} = -\nabla\vec{\phi} + Q}$$

Fourier's Law

$$\vec{\phi} = -K_0\nabla u$$

so the heat equation becomes

$$c\rho\frac{\partial u}{\partial t} = \nabla\cdot(K_0\nabla u) + Q$$

If $Q=0$, then

$$\frac{\partial u}{\partial t} = k\,\nabla\cdot(\nabla u), \quad k = \frac{K_0}{c\rho} \text{ (thermal diffusivity)}$$

Note

$$\nabla\cdot(\nabla u) = \nabla\cdot\left(\left\langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right\rangle\right)$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \cancel{\nabla^2 u \text{ or } \Delta u}$$

$\nabla^2 u$ or Δu (Laplacian of u)

Initial conditions:

$$u(x,y,z,0) = f(x,y,z)$$

Boundary conditions:

1. $u(x,y,z,t) = T(x,y,z,t)$ (prescribed temperature)

2. $-K_0 \nabla u \cdot \hat{n} = \vec{\phi}_{Bn}$ (prescribed normal heat flow)

   $\nabla u \cdot \vec{n} = 0$ (insulated condition)

3. Newton's Law of cooling

$$-K_0 \nabla u \cdot \hat{n} = H(u - u_b)$$

Steady state problem (Poisson's ~~problem~~ equation)

If the boundary conditions & sources are independent of time then a steady state solution may be found by solving

$$\boxed{0 = \nabla \cdot (K_0 \nabla u) + Q \quad \Rightarrow \quad \nabla^2 u = -\frac{Q}{K_0}}$$

This is called Poisson's ~~problem~~. equation

If $Q = 0$, $\boxed{\nabla^2 u = 0}$ (*) (laplacian of temp distribution is zero)

This is called the Laplace's equation.

(*) is also known as the potential equation because gravitational and electrostatic potential satisfy (*).

Polar and Cylindrical coordinates (skip for now)

Laplacian

$$\boxed{\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}}$$

It is useful to change from rectangular to circular cylindrical coordinates.

$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

$$\boxed{\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}}$$

Spherical coordinates

$x = \rho\sin\phi\cos\theta$
$y = \rho\sin\phi\sin\theta$
$z = \rho\cos\phi$

$$\boxed{\nabla^2 u = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2\frac{\partial u}{\partial \rho}\right) + \frac{1}{\rho^2\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial u}{\partial \phi}\right) + \frac{1}{\rho^2\sin^2\phi}\frac{\partial^2 u}{\partial \theta^2}}$$

Laplace's Equation inside a rectangle

1. Equilibrium temperature inside $0 \le x \le L$, $0 \le y \le H$.

Pde

$$\boxed{\begin{gathered}\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \\ u(0,y) = g_1(y), \quad u(L,y) = g_2(y), \quad u(x,0) = f_1(x) \quad u(x,H) = f_2(x).\end{gathered}}$$

Remarks

1. The pde is linear and homogeneous but the boundary conditions are non-homogeneous.
2. We can use the method of separation of variables with some modifications.

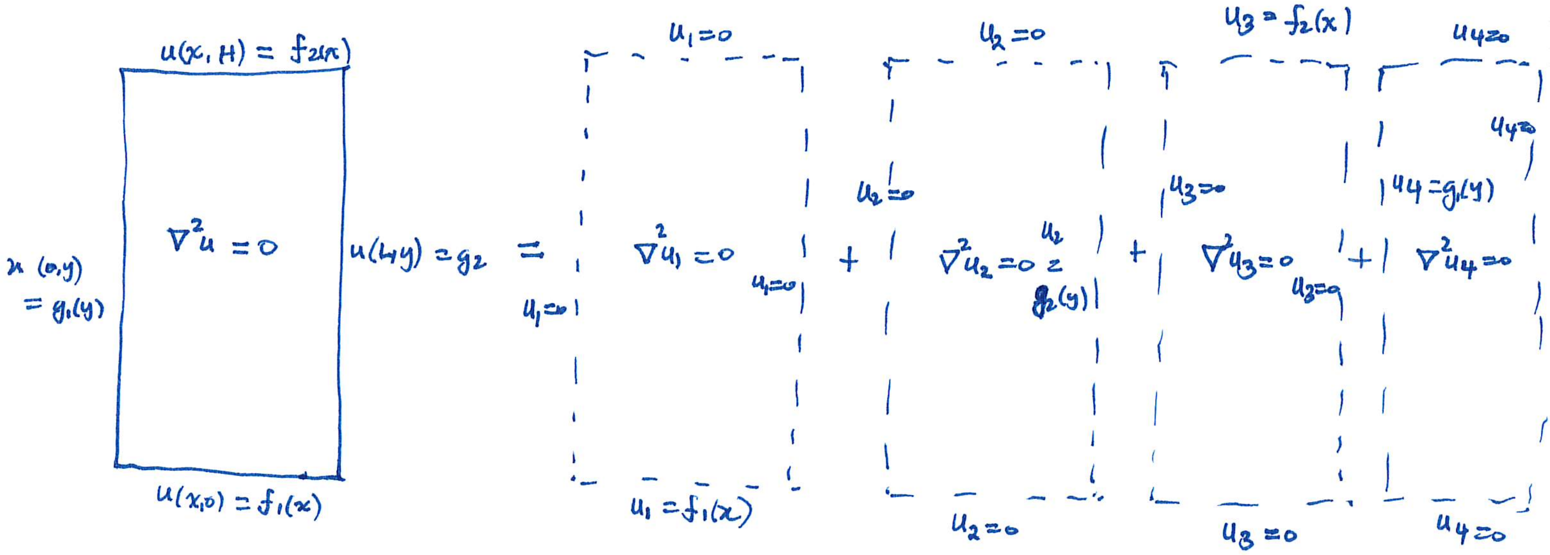
Strategy

Notice that the problem is non-homogeneous due to the 4 b.c so split up

$$u(x,y) = u_1(x,y) + u_2(x,y) + u_3(x,y) + u_4(x,y)$$

by the principle of superposition.

* Each $u_i(x,y)$ satisfies Laplace's equation with one non-homogeneous condition.



Solving for $u_4(x,y)$

$$\boxed{\begin{gathered}\frac{\partial^2 u_4}{\partial x^2} + \frac{\partial^2 u_4}{\partial y^2} = 0 \\ u_4(0,y) = g_1(y), \quad u_4(L,y)=0, \quad u_4(x,0) = 0, \quad u_4(x,H) = 0.\end{gathered}}$$

We use the method of separation of variables but ignore the ~~ho~~ non-homogeneous boundary condition.

$u_4(x,y) = h(x)\,\phi(y)$, then from the boundary conditions

$$u_4(L,y) = 0 \Rightarrow h(L) = 0$$

$$\boxed{\begin{aligned} u_4(x,0) &= 0 \Rightarrow \phi(0) = 0 \\ u_4(x,H) &= 0 \Rightarrow \phi(H) = 0.\end{aligned}}$$

the y component of the solution has 2 ~~non~~ homogeneous boundary conditions so this becomes the eigen value problem!

Plugging into $\frac{\partial^2 u_4}{\partial x^2} + \frac{\partial^2 u_4}{\partial y^2} = 0$, the form of the separated solution

$u(x,y) = h(x)\,\phi(y)$ yields

$$\phi(y)\frac{d^2 h}{dx^2} + h(x)\frac{d^2\phi}{dy^2} = 0 \Rightarrow \boxed{\frac{1}{h}\frac{d^2 h}{dx^2} = -\frac{1}{\phi}\frac{d^2\phi}{dy^2}}$$

divide by $\phi(y)\,h(x)$

* as before the only way to get this is if each side equals a constant.

$$\boxed{\frac{1}{h}\frac{d^2 h}{dx^2} = -\frac{1}{\phi}\frac{d^2\phi}{dy^2} = \lambda.}$$

We now solve 2 odes

$$\frac{d^2 h}{dx^2} = \lambda h$$

$$h(L) = 0$$

&

Eigenvalue problem

$$\frac{d^2\phi}{dy^2} = -\lambda\phi.$$

$$\phi(0) = 0$$

$$\phi(H) = 0$$

$$\frac{d^2\phi}{dy^2} = -\lambda\phi, \quad \phi(0) = 0, \quad \phi(H) = 0.$$

We have seen this problem before, the ~~eigenfu~~ eigenvalues are all positive

$$\lambda = \left(\frac{n\pi}{H}\right)^2, \quad n = 1, 2, 3, \ldots$$

and eigenfunctions are

$$\phi(y) = \sin\left(\frac{n\pi y}{H}\right).$$

Going back to the problem in x using $\lambda = \left(\frac{n\pi}{H}\right)^2$

$$\frac{d^2h}{dx^2} = \left(\frac{n\pi}{H}\right)^2 h \quad (*) \qquad h(L) = 0$$

The general solution to $(*)$ is

$$h(x) = a_1 \cosh\left(\frac{n\pi}{H}x\right) + a_2 \sinh\left(\frac{n\pi}{H}x\right)$$

Characteristic polynomial of $(*)$ is

$r^2 - \lambda h = 0$

$r = \pm\sqrt{\lambda}$

$h(x) = c_1 e^{-\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}$

$\cosh z = \frac{e^z + e^{-z}}{2}$

$\sinh z = \frac{e^z - e^{-z}}{2}.$

however this condition is not well suited to satisfy the boundary condition $h(L) = 0$.

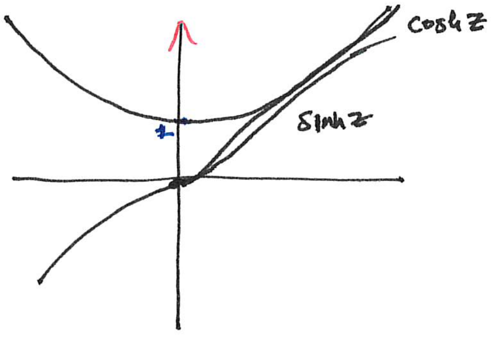
We introduce a translation $x' = x - L$, this does not change the ODE because of the chain rule.

so we adopt a translated solution

$$h(x) = a_1 \cosh\left(\frac{n\pi}{H}(x - L)\right) + a_2 \sinh\left(\frac{n\pi}{H}(x - L)\right)$$

this still solves the original problem

as the general solution.



$$h(L) = 0 \Rightarrow a_1 \cosh\left(\frac{n\pi}{H}(x - L)\right) = 0$$

so $a_1 = 0$ $\quad (\cosh(0) = 1)$.

Recall that

$$\cosh z = \frac{e^z + e^{-z}}{2}$$

$$\sinh z = \frac{e^z - e^{-z}}{2}.$$

so $$h(x) = a_2 \sinh\left(\frac{n\pi}{H}(x-L)\right)$$

The product solution

$$u_4(x,y) = h(x)\,\phi(y)$$
$$= \underbrace{\underbrace{A \sin\left(\frac{n\pi y}{H}\right)}_{\text{oscillatory solution}} \underbrace{\sinh\left(\frac{n\pi}{H}(x-L)\right)}_{\text{non-oscillatory solution.}}}_{\text{This turns out to be a general property of Laplace equation.}}$$

By the principle of superposition

$$u_4(x,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi y}{H}\right) \sinh\left(\frac{n\pi}{H}(x-L)\right)$$

We want

$$u_4(0,y) = g_1(y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi y}{H}\right) \sinh\left(\frac{n\pi}{H}(-L)\right).$$

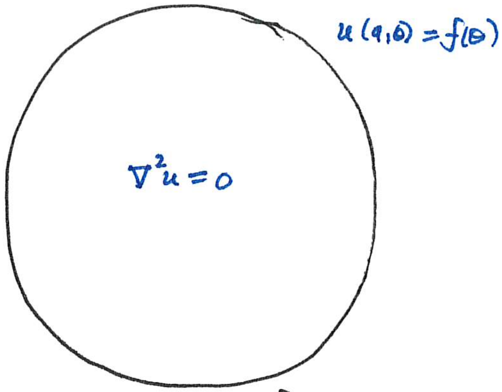
Using the orthorgonality of $\left\{\sin\left(\frac{n\pi y}{H}\right)\right\}$

$$\int_0^H g_1(y) \sin\left(\frac{m\pi y}{H}\right) dy = \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi(-L)}{H}\right) \int_0^H \sin\left(\frac{n\pi y}{H}\right) \sin\left(\frac{m\pi y}{H}\right) dy$$
$$= A_m \sinh\left(\frac{m\pi(-L)}{H}\right) \cdot \frac{H}{2}$$

$$A_m = \frac{2}{H\left(\sinh\left(\frac{n\pi}{H}(-L)\right)\right)} \int_0^H g_1(y) \sin\left(\frac{m\pi y}{H}\right) dy.$$

repeat for u_1, u_2, u_3 ...

Laplace's equation in a disk



- circular disk with constant thermal properties
- prescribed temperature on the boundary

The geometry dictates switching from rectangular to polar via

$x = r\cos\theta$
$y = r\sin\theta$

then

$$\boxed{\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.}$$

Boundary conditions

$$u(a,\theta) = f(\theta). \qquad (1)$$

* At first glance due to $u(a,\theta) = f(\theta)$ [a non-homogeneous condition], it appears that we cannot use separation of variables.

** In the case of rectangular coordinates we specified conditions at the end points of $0 \le x \le L$ and $0 \le y \le H$.

*** We can do the same for polar coordinates

$$0 \le r \le a \quad , \quad -\pi \le \theta \le \pi$$

$r = 0$

The temperature is finite, $|u(0,\theta)| < \infty$. (2)

at $\theta = \pm\pi$

Periodic boundary conditions (temperature and heat flow is continuous in the θ-direction)

$$u(r,-\pi) = u(r,\pi) \qquad (3)$$

$$\frac{\partial u}{\partial \theta}(r,-\pi) = \frac{\partial u}{\partial \theta}(r,\pi) \qquad (4)$$

Conditions (2)–(4) are all linear and homogeneous, indeed $u \equiv 0$ satisfies all of them.

In summary, now we have

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \qquad (*)$$

$$u(a,\theta) = f(\theta) \qquad (1)$$

$$|u(0,\theta)| < \infty \qquad (2)$$

$$u(r,-\pi) = u(r,\pi) \qquad (3)$$

$$\frac{\partial u}{\partial \theta}(r,-\pi) = \frac{\partial u}{\partial \theta}(r,\pi) \quad (4)$$

We can use separation of variables because only (1) is non-homogeneous.

Indeed, $\boxed{\text{Let } u(r,\theta) = \phi(\theta)G(r).}$ (product solutions of θ and r)

$$u(r,-\pi) = u(r,\pi) \Rightarrow \phi(-\pi) = \phi(\pi)$$

$$\frac{\partial u}{\partial \theta}(r,-\pi) = \frac{\partial u}{\partial \theta}(r,\pi) \Rightarrow \frac{d\phi}{d\theta}(-\pi) = \frac{d\phi}{d\theta}(\pi).$$

Plugging in the product solution into the pde $(*)$ yeilds

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dG}{dr}\right)\phi(\theta) + \frac{1}{r^2}G(r)\frac{d^2\phi}{d\theta^2} = 0$$

Seperate variables by dividing by $\left(\frac{1}{r^2}\right)G(r)\phi(\theta)$ so that

$$\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = -\frac{1}{\phi}\frac{d^2\phi}{d\theta^2} = \lambda.$$

(Note: we pick the separation constant λ (rather than $-\lambda$) to ensure that the θ problem has oscillatory solutions)

Boundary value problem

$$\frac{d^2\phi}{d\theta^2} = -\lambda\phi$$

$$\phi(-\pi) = \phi(\pi)$$

$$\frac{d\phi}{d\theta}(-\pi) = \frac{d\phi}{d\theta}(\pi)$$

We have a circular wire with $L = \pi$

Compare to the eigenvalue problem for the circular ring!

$$\frac{d^2\phi}{dx^2} = -\lambda\phi$$

$$\phi(-L) = \phi(L)$$

$$\frac{d\phi}{dx}(-L) = \frac{d\phi}{dx}(L)$$

**

So

$$\lambda = \left(\frac{n\pi}{L}\right)^2 = n^2.$$

with eigenfunctions $\sin(n\theta)$ and $\cos(n\theta)$

r - dependent problem

$$\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = \lambda = n^2$$

$$\frac{r}{G}\left[r\frac{d^2G}{dr^2} + \frac{dG}{dr}\right] = n^2$$

$$r^2\frac{d^2G}{dr^2} + r\frac{dG}{dr} = n^2 G$$

This yields the following problem

$$\boxed{\begin{array}{l} r^2\dfrac{d^2G}{dr^2} + r\dfrac{dG}{dr} - n^2 G = 0 \\ |u(0,\theta)| < \infty \Rightarrow |G(0)| < \infty . \end{array}} \qquad (*)$$

Notice that $G = r^p$ reproduces itself.

Plug in $G = r^p$ into $(*)$ so that

$$r^2\left[p(p-1)r^{p-2}\right] + r\left[pr^{p-1}\right] - n^2 r^p = 0$$

$$\left[p(p-1) + p - n^2\right]r^p = 0$$

$$\left[p^2 - p + p - n^2\right]r^p = 0 \Rightarrow p^2 = n^2 \Rightarrow p = \pm n, \quad n \neq 0$$

the general solution takes the form

$$G = c_1 r^n + c_2 r^{-n}$$

$n=0$

$r^0 = 1$ is a solution.

$$\frac{d}{dr}\left(r\frac{dG}{dr}\right) = 0 \Rightarrow r\frac{dG}{dr} = k \Rightarrow \frac{dG}{dr} = \frac{k}{r}$$

so the second solution is $\ln(r)$

$$G = \bar{c}_1 + \bar{c}_2 \ln(r)$$

Boundary condition

We need. $|G(0)| < \infty$

$$G(r) = c_1 r^n + c_2 r^{-n} \quad \& \quad G(r) = \bar{c}_1 + \bar{c}_2 \ln(r)$$

We need in each case

$$\lim_{r\to 0} G(r) \not< \infty$$

which is only true if $c_2 = \bar{c}_2 = 0$.

The solution of interest (bounded at the origin) is

$$G(r) = c_1 r^n, \quad n \geq 0.$$

This yields product solutions

$$r^n \cos(n\theta) \quad (n \geq 0) \quad \text{and} \quad r^n \sin(n\theta) \quad (n \geq 1)$$

so by the principle of superposition

$$u(r,\theta) = \sum_{n=0}^{\infty} A_n r^n \cos(n\theta) + \sum_{n=1}^{\infty} B_n r^n \sin(n\theta), \quad \begin{array}{l} 0 \leq r < a \\ -\pi < \theta \leq \pi. \end{array}$$

back to the non-homogeneous condition:

$$u(a,\theta) = f(\theta).$$

from our general solution,

$$u(a,\theta) = \sum_{n=0}^{\infty} A_n a^n \cos(n\theta) + \sum_{n=1}^{\infty} B_n a^n \sin(n\theta), \quad -\pi < \theta \le \pi.$$

From the orthogonality of $\sin(n\theta)$ and $\cos(m\theta)$, we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta = A_0$$

$n \geq 1$ $$A_n = \frac{1}{a^n\pi}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\,d\theta$$

$$B_n = \frac{1}{a^n\pi}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)\,d\theta.$$