Introduction

GOAL – derivation + solution techniques for elementary partial differential equations (PDEs)

PDE

A pde is an identity that relates a dependent variable $u(t, x, y, \dots)$ and the partial derivatives of u, here $t, x, y, \dots$ are independent variables.

An ode has one variable.

Examples

(i) $\dfrac{\partial u}{\partial t} + 3\dfrac{\partial u}{\partial x} = 0$ (Simple linear PDE)

(ii) $\dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} = 0$ (1D Heat Equation)

(iii) $\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial y^2} = 0$ (Wave Equation)

(iv) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = g(x,y)$ (Poisson's Equation)

(v) $\underbrace{\rho}_{\text{mass } \rho \text{ of fluid}}\left(\underbrace{\dfrac{\partial v}{\partial t} + v\cdot\nabla v}_{\text{accelaration}}\right) = \underbrace{\overset{\text{internal pressure}}{\nabla p} + \overset{\text{external f}}{\rho g} + \overset{\text{internal stress}}{\mu \nabla^2 v}}_{\text{force}}$ (Navier – Stokes)

QUESTIONS

1. Find solutions $u(x,t)$ (in the case g(i, ii)) or $u(x,y)$ for (iii) and (iv) & $u(t,x,y,z)$ for (v) satisfying the PDEs.

2. Understand the underlying physical problems. ⇒ $\boxed{\text{Chemistry, Physics; Geophysics; Elasticity etc}}$

GENERAL STRATEGY

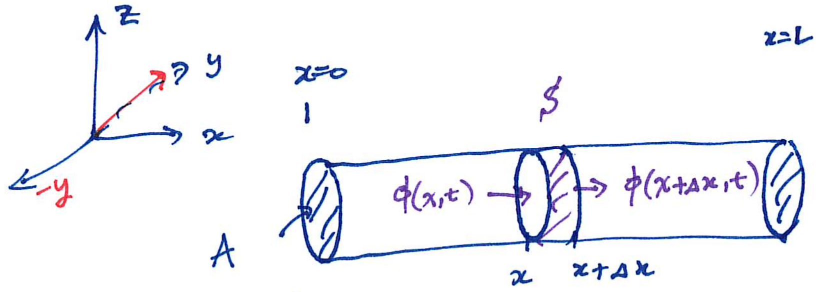
Formulation → Solution → Interpretation

HEAT EQUATION (1D)

– Thermal energy moves due to (1) conduction (collisions of neighboring molecules resulting in transfer of kinetic energy) OR convection (molecules move from one region to another).

Conduction of heat in a 1D rod

* (conduction dominates convection)



- Rod with constant cross-sectional area A, $0 \le x \le L$.

I $\boxed{\text{THERMAL ENERGY DENSITY}}$

Let $e(x,t) \equiv$ thermal energy density (constant across a cross section A) so that the rod is one dimensional in x at time t.

Physically this can be achieved by insulating the lateral surface area.

$e(x,t) \Rightarrow$ The rod is not uniformly heated. $\Rightarrow$ Go to II HEAT ENERGY

~~III HEAT ENERGY~~

III $\boxed{\text{HEAT FLUX}}$

Assuming thermal energy flows from left to right.

$\phi(x,t) \equiv$ heat flux (amount of energy per unit time flowing to the right per unit surface area).

Convention $\phi(x,t) < 0 \Rightarrow$ heat is flowing to the left.

$$\boxed{\phi(x,t)A - \phi(x+\Delta x,t)A}$$ $\Rightarrow$ Heat flowing per unit time across the boundaries of the slice of thickness Δx.

If $\phi(x,t) > 0$ and $\phi(x+\Delta x,t) > 0 \Rightarrow$ heat flow @ x causes an increase in heat energy in S and $\phi(x+\Delta x,t)$ causes a decrease in energy in S

III. HEAT ENERGY

For small Δx, $e(x,t)$ is constant in ΔS so

$$\boxed{\text{heat energy} = e(x,t)\,A\,\Delta x.}$$

(Energy density $\times$ Volume)

IV $\boxed{\text{HEAT SOURCES}}$

Heat from external sources. $\boxed{\text{eg electrical heating, chemical reactions}}$.

$Q(x,t) \equiv$ heat energy per unit volume per unit time.

eg

For small enough Δx, $Q(x,t)$ is approximately constant so

$$\boxed{\text{Total energy generated per unit time} = Q(x,t)\,A\Delta x}$$

Conservation of Heat Energy (thin slice)

$\boxed{\text{Rate of change of heat Energy in time}}$ = Heat energy flowing across the boundary per unit time. + Heat energy generated inside per unit time.

$$\boxed{\frac{\partial}{\partial t}\left(e(x,t)\,A\Delta x\right) \simeq \phi(x,t)A - \phi(x+\Delta x,t)A + Q(x,t)A\Delta x} \quad (1).$$

If we divide (1) by $A\Delta x$ and take $\lim_{\Delta x \to 0}$,

$$\frac{\partial}{\partial t}\left(e(x,t)\right) = \lim_{\Delta x \to 0}\left(\frac{\phi(x,t) - \phi(x+\Delta x,t)}{\Delta x}\right) + Q(x,t).$$

yeilding

$$\boxed{\frac{\partial e}{\partial t} = -\frac{\partial \phi}{\partial x} + Q(x,t)} \quad (2) \quad \text{[CONSERVATION OF HEAT energy]}$$

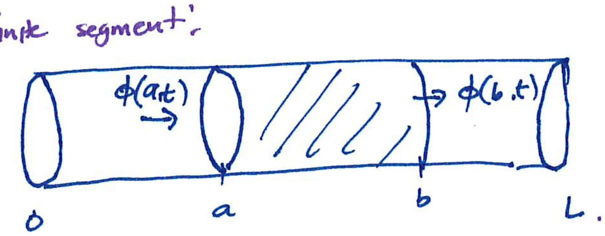
Ignoring Q,

If $\frac{\partial \phi}{\partial x} > 0$, ϕ is increasing in x so the heat energy must decrease.

Alternative approach (Integral approach).

EXACT CONSERVATION OF HEAT ENERGY (EXACT).

Consider a finite segment:



Conservation of energy on the segment $a \le x \le b$.

recall that $e(x,t)$ is the thermal energy density.

$$\boxed{\text{TOTAL HEAT ENERGY} = \int_a^b e(x,t)\,A\,dx}$$

(Sum of contributions of infinitesimal slices)

Energy per unit ~~length~~ infinitesimal slice $= e(x,t)\,A\,\Delta x$.

$$\lim_{n\to\infty} \sum_{i=1}^{n} e(x_i,t)\,A\,\Delta x$$

$$\boxed{\text{Heat flowing through boundaries per unit time} = A\phi(a,t) - A\phi(b,t)}$$

total heat generated $\int_a^b Q(x,t)\,A\,dx$.

$Q(x,t)$ – heat energy / unit vol / unit time.

Rate of change of heat energy in / unit time

so

$$\frac{d}{dt} A\int_a^b e(x,t)\,dx = A\,\phi(a,t) - A\,\phi(b,t) + A\int_a^b Q(x,t)\,dx$$

Heat flowing across the boundary / unit time

Heat energy generated inside / unit time

Yielding

divide by A

$$\frac{d}{dt}\int_a^b e(x,t)\,dx = \phi(a,t) - \phi(b,t) + \int_a^b Q(x,t)\,dx$$

(Integral conservation Law)

Notice that

① $$\frac{d}{dt}\int_a^b e(x,t)\,dx = \int_a^b \frac{\partial e}{\partial t}\,dx$$ provided $e(x,t)$ is continuous.

full derivative

partial derivative (x is fixed).

② By FTC,

$$\phi(a,t) - \phi(b,t) = -\int_a^b \frac{\partial \phi}{\partial x}\,dx$$

* if $\phi(x,t)$ is continuously differentiable.

then

$$\int_a^b \left(\frac{\partial e}{\partial t} + \frac{\partial \phi}{\partial x} - Q\right) dx = 0.$$ for any a and b

so $$\frac{\partial e}{\partial t} + \frac{\partial Q}{\partial x} - Q = 0 \Rightarrow \boxed{\frac{\partial e}{\partial t} = -\frac{\partial \phi}{\partial x} + Q}$$

Significance of minus sign in

$$\boxed{\frac{\partial e}{\partial t} = -\frac{\partial \phi}{\partial x} + Q.}$$

If $\frac{\partial \phi}{\partial x} > 0$, ϕ is increasing in x so heat is flowing greater at $x=b$ than $x=a$ so the heat ~~no~~ energy must decrease between $x=a$ and $x=b$.

TEMPERATURE and SPECIFIC HEAT

$u(x,t)$ - Temperature.

is a more natural variable to describe material heat properties. $\Rightarrow$ we want to write e and ϕ in terms of u.

Specific heat , $c(x)$, or c if the rod is made of uniform material

The heat that must be supplied to a unit of mass of a substance to raise its temperature by one unit

Thermal energy = $e(x,t) A \Delta x$ (thermal energy density x Volume).

This is also defined as the energy it takes to raise temperature from a reference temp (0°) to $u(x,t)$ so

"Thermal energy
heat energy per unit mass = $c(x)\, u(x,t)$

so if $\rho(x)$ is the mass density $\left({}^{\text{mass}}/_{\text{unit volume}}\right)$

Thermal energy = (Thermal energy / unit mass) x mass.

$$e(x,t) A \Delta x = c(x)\, u(x,t) \rho A \Delta x.$$

mass $= \rho \cdot V$
$= \rho A \Delta x$.

so

$$\boxed{e(x,t) = c(x)\, \rho(x)\, u(x,t)}$$

so the energy conservation law becomes

$$c(x)\, \rho(x) \frac{\partial u}{\partial t} = -\frac{\partial \phi}{\partial x} + Q$$

Now, for the Heat flux...

Dependence of heat flux ($\phi(x,t)$) on temperature.

FOURIER'S LAW :

$$\boxed{\phi = -K_0 \frac{\partial u}{\partial x}.}$$

$\frac{\partial u}{\partial x}$ – rate of change of u (fixed t).

Heat flux is proportional to temperature difference

- If u is constant → no heat flows. • heat flows from hot to cold.
- the larger the temperature differences the greater the flow
- Flow depends on the material thermal conductivity K_0 → large K_0 ⇒ greater conductivity.

Negative sign

Suppose u increases from left to right, $\frac{\partial u}{\partial x} > 0$, then $\phi < 0$ so heat flows to the left!

HEAT EQUATION

$$c\rho \frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\left(-K_0 \frac{\partial u}{\partial x}\right) + Q$$

$$c\rho \frac{\partial u}{\partial t} = K_0 \frac{\partial^2 u}{\partial x^2} + Q.$$

If there are no sources, $Q = 0$,

$$\boxed{\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}} \qquad (D)$$

where $k = \frac{K_0}{c\rho}$ is called the thermal diffusivity

(D) is also called the diffusion equation.

- If heat is initially concentrated in one place, (D) describes diffusion.
- $u(x,t)$ could also concentration of pollutants / chemicals.

Initial conditions.

Recall that when solving ODEs with one derivative we need an initial condition

eg $$u'(t) = f(t).$$
$$u(0) = \alpha$$

2 derivatives need 2 initial conditions

eg $$u''(t) = f(t)$$
$$u'(0) = \alpha$$
$$u(0) = \beta$$

We do the same for

$$\boxed{\frac{du}{dt} = k\frac{d^2u}{dx^2}}$$

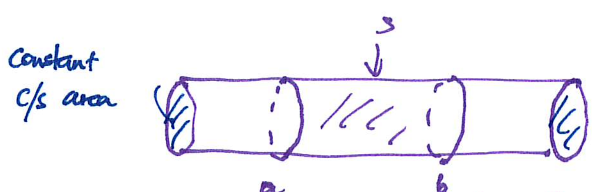
- We have one time derivative so we need one (IC) eg @ $t=0$
  $u(x,0) = f(x)$ (initial temperature distribution).

- 2 spatial derivatives so we need 2 conditions so we specify 2 boundary conditions.

  Once these 3 conditions are specify we can solve the heat equation to predict the future temperature distribution at any time t.

Diffusion of a chemical pollutant

Let $u(x,t)$ denote the concentration of a chemical per unit volume / density



Total amount of chemical in S = chemical density × Volume.

$$= \sum_a^b u(x,t) \times \text{Area} \times \Delta x. \quad , \Delta x \to 0 \text{ (summed)}.$$

as an integral $$\int_a^b u(x,t)\, A\, dx$$

Let $\phi(x,t)$ be the flux of the chemical

$\phi(x,t)$ – amount of chemical per unit surface area per unit time flowing to the right.

Integral Conservation Law ($Q=0$)

$$\frac{d}{dt}\int_a^b u(x,t)\, A\, dx = A\big(\phi(a,t) - \phi(b,t)\big)$$

dividing both sides by A and using the fact that

$$\frac{d}{dt}\int_a^b u(x,t)\,dx = \int_a^b \frac{\partial u}{\partial t}\,dx \quad \text{and} \quad \phi(a,t) - \phi(b,t) = -\int_a^b \frac{\partial \phi}{\partial x}\,dx$$

$$\int_a^b \left(\frac{\partial u}{\partial t} + \frac{\partial \phi}{\partial x}\right) dx = 0 \quad \text{for any value of } a, b \text{ so}$$

$$\boxed{\frac{\partial u}{\partial t} + \frac{\partial \phi}{\partial x} = 0}$$

Fick's law of diffusion $\quad \phi \propto \frac{\partial u}{\partial x} \quad$ so $\quad \phi = -k\frac{\partial u}{\partial x}$

so

$$\boxed{\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}}$$

BOUNDARY CONDITIONS

(1) PRESCRIBED TEMPERATURE (DIRICHLET BOUNDARY CONDITION)

$$u(0,t) = u_B(t) \quad (\text{prescribed temperature @ } x=0).$$

$u_B(t)$ is the temperature of a fluid bath or reservior.

(2) INSULATED BOUNDARY (homogeneous Neumann Boundary Condition).

Insulated at $x=0$

$$-K_0 \frac{\partial u}{\partial x}(0,t) = 0. \quad (\text{No heat flow}).$$

(2b) the flux (non-zero) may also be prescribed

$$-K_0(0)\frac{\partial u}{\partial x}(0,t) = \phi_B(t)$$

(3) NEWTON'S LAW OF COOLING..

(One dimensional rod in contact with a moving fluid).

Heat flow is proportional to the difference between the bar and the external temperature.

$$-K_0(0)\frac{\partial u}{\partial x}(0,t) = -H[u(0,t) - u_B(t)]. \quad (1)$$

Fourier's Law

$$\phi = -K_0 \frac{\partial u}{\partial x}$$

where $H^{>0}$ is the heat transfer coefficient.

$\phi < 0$

- If $u(0,t) > u_B(t)$ then heat flows out of the rod. (left flow $\frac{\partial u}{\partial x} > 0$)
- If $u(0,t) < u_B(t)$ then the rod is cooler than the surrounding ~~air~~ bath / fluid so the heat flux > 0.

Remarks

(1) H is experimentally determined (depends on rod and fluid)

As $H \to 0$, Newton's law (1) approaches the insulated boundary condition.

(2) As $H \to \infty$, $-\frac{K_0(0)}{H}\frac{\partial u}{\partial x}(0,t) = -[u(0,t) - u_B(t)]$, then Newton's Law of

cooling approaches the prescribed temperature condition.

$$u(0,t) = u_B(t).$$