Chapter 2

Method of Separation of Variables

GOAL: obtain solutions to the heat equation of the form

$$u(x,t) = \phi(x)\, G(t). \quad \text{(separated solution)}.$$

where $\phi(x)$ is a function of x and $G(t)$ is a function only in t.

This method is used when the PDE & boundary conditions

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + \frac{Q(x,t)}{c\rho}, \quad \begin{matrix} t > 0 \\ 0 < x < L \end{matrix}$$

$$u(x,0) = f(x)$$

$$u(0,t) = T_1(t), \quad u(L,t) = T_2(t)$$

is are linear and homogeneous

LINEARITY

An ode operator L satisfies the property

$$L(c_1 u_1 + c_2 u_2) = c_1 L(u_1) + c_2 L(u_2)$$

where u_1 and u_2 are functions and c_1 and c_2 are constants.

HEAT OPERATOR

$$L(u) = \frac{\partial u}{\partial t} - k \frac{\partial^2 u}{\partial x^2}$$

is a linear operator.

PROOF

$$L(c_1 u_1 + c_2 u_2) = \frac{\partial}{\partial t}(c_1 u_1 + c_2 u_2) - k \frac{\partial^2}{\partial x^2}(c_1 u_1 + c_2 u_2)$$

$$= c_1 \frac{\partial u_1}{\partial t} + c_2 \frac{\partial u_2}{\partial t} - k c_1 \frac{\partial^2 u_1}{\partial x^2} - k c_2 \frac{\partial^2 u_2}{\partial x^2}$$

$$= c_1 \left(\frac{\partial u_1}{\partial t} - k \frac{\partial^2 u_1}{\partial x^2} \right) + c_2 \left(\frac{\partial u_2}{\partial t} - k \frac{\partial^2 u_2}{\partial x^2} \right)$$

$$= c_1 L u_1 + c_2 L u_2 \qquad \square$$

We can write our pde in functional notation as

$$\mathcal{L}(u) = \frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = f(x,t).$$

If $f(x,t) = 0 \Rightarrow L(u) = 0$ the problem is called a linear homogeneous equation.

Remarks

1. $u = 0$ is the trival solution of $L(u) = 0$

If $f \neq 0$ the equation is non-homogeneous.

Principal of Superposition

The linear property allows the following:

If u_1 and u_2 satisfy the linear homogeneous equation, then an arbitrary linear combination, $c_1 u_1 + c_2 u_2$ also satisfies the same linear homogeneous equation.

PROOF

u_1 and u_2 are homogeneous solutions, therefore $L(u_1) = 0$ & $L(u_2) = 0$

Compute

$$L(c_1 u_1 + c_2 u_2) \underset{\uparrow}{=} c_1 L(u_1) + c_2 L(u_2) = c_1 \cdot 0 + c_2 \cdot 0$$

linearity property

therefore $c_1 u_1 + c_2 u_2$ are solutions to the homogeneous problem $L(u) = 0$.

The concepts of linearity and homogeneity also apply to boundary conditions.

$$\left.\begin{aligned} u(0,t) &= f(t) \\ \frac{\partial u}{\partial x}(L,t) &= g(t) \end{aligned}\right\} \text{ non-homogeneous}$$

$u(0,t) = 0$, $\frac{\partial u}{\partial x}(L,t) = 0$ are homogeneous.

HEAT EQUATION WITH HOMOGENEOUS DIRICHLET CONDITIONS

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \quad , \quad 0 < x < L \, , \quad t > 0$$

$$(P).$$

B.C:

$$u(0,t) = 0 \quad , \quad u(L,t) = 0$$

Initial condition:

$$u(x,0) = f(x).$$

Remarks (Why P?)

1. We can use the method of separation of variables to solve (P)
2. Relavant physical problem - 1D rod with ends immersed in a bath of temp $0°$.
3. Solving the non-homogeneous problem requires knowing how to solve the homogeneous problem.

METHOD (Separation of variables)

1. Plug in the form of the solution

$$u(x,t) = \phi(x)\, G(t)$$

into (P) while ignoring the initial conditions for now.

$$\frac{\partial}{\partial t}\Big(\phi(x)\, G(t)\Big) = k \frac{d}{dx^2}\Big(\phi(x)\, G(t)\Big)$$

$$\phi(x) \frac{dG}{dt} = k \frac{d^2\phi}{dx^2} G(t).$$

divide both sides by $k\, \phi(x)\, G(t)$ to separate the variables

$$\underbrace{\frac{1}{kG}\frac{dG}{dt}}_{\text{function only in } t} = \underbrace{\frac{1}{\phi}\frac{d^2\phi}{dx^2}}_{\text{function only in } x.}$$

The only way we can have a function of t to be equal to a function in x (t and x are independent) is if both sides are equal to a constant.

$$\frac{1}{kG}\frac{dG}{dt} = \frac{1}{\phi}\frac{d^2\phi}{dx^2} = -\lambda.$$

where λ is the separation constant.

Separation yeilds

$$\frac{d^2\phi}{dx^2} = -\lambda\phi \quad \text{and} \quad \frac{dG}{dt} = -\lambda k G$$

$u(x,t) = \phi(x)\,G(t)$ must also satisfy the boundary conditions $u(0,t) = 0$ & $u(L,t) = 0$.

i.e $u(0,t) = 0 \Rightarrow \phi(0)\,G(t) = 0 \Rightarrow$ either $\phi(0) = 0$ OR $G(t) \equiv 0$

If $G(t) \equiv 0$, then $u(x,t) \equiv 0$ is a trivial solution. We are interest in non-trivial solutions! It must be the case that $\phi(0) = 0$.

Similarly $\phi(L) = 0$.

Time dependent ODE

$$\frac{dG}{dt} = -\lambda k G$$ (first-order linear homogeneous ODE w constant coefficients)

We seek exponential solutions of the form $G = e^{rt}$

the characteristic polynomial is

$r = \lambda k$. So the general solution is $\boxed{G(t) = c e^{-\lambda k t}}$

λ is an arbitrary separation constant.

1. If $\lambda > 0$, $G(t) \to 0$ as $t \to \infty$
2. If $\lambda < 0$, $G(t) \to \infty$ as $t \to \infty$
3. If $\lambda = 0$, $G(t)$ is constant.

$\Rightarrow$ since $u(x,t) = \phi(x)\,G(t)$ is the temperature of the rod, we can eliminate case 2. $\Rightarrow$ $\boxed{\lambda \geq 0}$

This is the reason we picked $-\lambda$.

Allowable separation constants (BOUNDARY VALUE PROBLEM)

We solve the Boundary value problem (Eigenvalue problem)

$$\boxed{\begin{array}{l}\dfrac{d^2\phi}{dx^2} = -\lambda\phi \\ \phi(0) = 0, \quad \phi(L) = 0.\end{array}} \qquad (P)$$

- There is no simple theory that determines when the solution to (P) exists.
- It is clear that $\phi(x) = 0$ is a trivial solution – the key question is for what values of λ ~~are~~ can we have non-trival solutions?

KEY (Eigenvalues)

1. There exist certain values of λ for which non-trival solutions exist. These values are called eigenvalues. of the boundary value (P).
2. The non-trival solutions corresponding to the eigenvalues λ are called eigenfunctions.

We solve $\phi''(x) = \lambda\phi(x)$ by seeking solutions of the form

$\phi(x) = e^{rx} \Rightarrow \phi''(x) = r^2 e^{rx}$

Recall the method of the characteristic polynomial

$$r^2 e^{rx} = -\lambda e^{rx} \Rightarrow r^2 = -\lambda \quad \text{because } e^{rx} \neq 0$$

Cases

1. $\lambda > 0 \Rightarrow$ 2 roots are purely imaginary, $r = \pm i\sqrt{\lambda}$ ✓
2. $\lambda = 0 \Rightarrow$ 2 roots coalesce, $r = 0$ (twice). ✓
3. $\lambda < 0 \Rightarrow$ 2 distinct real roots $r = \pm\sqrt{-\lambda}$ ✗
4. λ is complex. ✗

We will prove that $\lambda \in \mathbb{R}$ so we can discount 4.
Physical ~~that~~ intuition $\Rightarrow \lambda \geq 0$, we will prove that Case 3 also needs to go.

Case I : $\lambda > 0$ (Eigenvalues and eigenfunctions)

$$\frac{d^2\phi}{dx^2} = -\lambda\phi$$

$$\phi(0) = 0 \quad , \quad \phi(L) = 0.$$

If $\lambda > 0$, the exponential solutions are $\phi = e^{rx}$ satisfying $r^2 = -\lambda \Rightarrow r = \pm i\sqrt{\lambda}$

so $\phi(x) = e^{\pm i\sqrt{\lambda}x}$. (these solutions are oscillatory)

Since we seek real independent solutions, the choices $\cos(\sqrt{\lambda}x)$ and $\sin(\sqrt{\lambda}x)$ are usually made.

Note : $\cos(\sqrt{\lambda}x)$ and $\sin(\sqrt{\lambda}x)$ are linear combinations of $e^{\pm i\sqrt{\lambda}x}$ from Euler's formula

$$e^{ix} = \cos x + i\sin(x)$$

as any linear combination is a solution! (principal of superposition.

$$\frac{1}{2}\left(e^{i\sqrt{\lambda}x} + e^{-i\sqrt{\lambda}x}\right) = \cos(\sqrt{\lambda}x)$$

$$\frac{1}{2i}\left(e^{i\sqrt{\lambda}x} - e^{-i\sqrt{\lambda}x}\right) = \sin(\sqrt{\lambda}x)$$

The general solution is thus

$$\boxed{\phi(x) = c_1\cos(\sqrt{\lambda}x) + c_2\sin(\sqrt{\lambda}x)}$$

* $\cos(\sqrt{\lambda}x)$ and $\sin(\sqrt{\lambda}x)$ are more convenient but $e^{i\sqrt{\lambda}x}$ and $e^{-i\sqrt{\lambda}x}$ can be used.

** eigenvalue comes from the German word eigenwert meaning characteristic value.

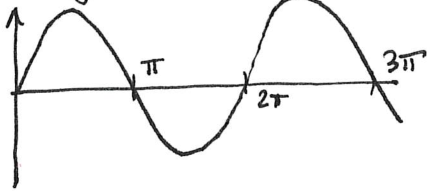
*** Other independent solutions may be chosen e.g $\cos\sqrt{\lambda}(x-a)$ and $\sin\sqrt{\lambda}(x-a)$

Applying boundary conditions

$$\phi(0) = 0 \Rightarrow c_1 = 0.$$

$$\phi(L) = 0 \Rightarrow c_2\sin(\sqrt{\lambda}L) = 0$$ (If $c_2 = 0$, then $\phi(x) \equiv 0$ this is a trivial solution)

This means that the eigenvalues should satisfy $\sin\sqrt{\lambda}L = 0 \Rightarrow \sqrt{\lambda}L = n\pi$, $n > 0$ as $\sqrt{\lambda} > 0$



The eigenvalues are

$$\lambda = \left(\frac{n\pi}{L}\right)^2 , \quad n = 1, 2, 3, \ldots$$

The eigen functions are

$$\phi(x) = c_2 \sin\sqrt{\lambda}\, x = c_2 \sin\left(\frac{n\pi x}{L}\right)$$ where c_2 is an arbitrary constant.

We can pick a convienient value of c_2 e.g 1.

$$\Rightarrow \quad \boxed{\phi(x) = c_2 \sin\left(\frac{n\pi x}{L}\right)}$$

$\boxed{\lambda \equiv 0}$

Is $\lambda = 0$ is an eigenvalue of

$$\frac{d^2\phi}{dx^2} = -\lambda\phi$$

$$\phi(0) = 0, \quad \phi(L) = 0 \quad ?$$

If $\lambda = 0$, $\phi''(x) = 0 \Rightarrow \phi = c_1 + c_2 x$.

Applying boundary conditions

$\phi(0) = 0 \Rightarrow c_1 = 0$ and $\phi(L) = 0 \Rightarrow c_2 L = 0 \Rightarrow c_2 = 0$ as $L > 0$ so $c_2 = 0$. This means that $\phi(x) \equiv 0$ a trivial solution!

Note: that $\lambda = 0$ may be an eigenvalue for other problems. One always has to check

$\boxed{\lambda < 0}$

If $\lambda < 0$ then the solution of

$$\frac{d^2\phi}{dx^2} = -\lambda\phi$$

is of the form $\phi(x) = e^{rx}$, where $r = \pm\sqrt{-\lambda}$, Let $-\lambda = s$, $s > 0$

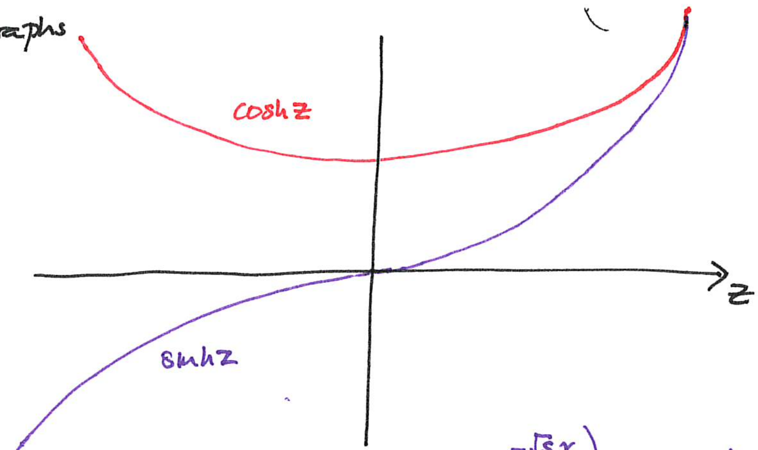
So that $\dfrac{d^2\phi}{dx^2} = s\phi$ with 2 independent solutions $e^{\sqrt{s}x}$ and $e^{-\sqrt{s}x}$ with a general solution

$$\boxed{\phi(x) = c_1 e^{\sqrt{s}x} + c_2 e^{-\sqrt{s}x}}$$

Recall the hyperbolic functions

$$\cosh z \equiv \frac{e^{z}+e^{-z}}{2} \quad \text{and} \quad \sinh z \equiv \frac{e^{z}-e^{-z}}{2}$$

with graphs



Using the hyperbolic functions (instead of $e^{-\sqrt{s}x}$) we have

$$\boxed{\phi(x) = c_3 \cosh\sqrt{s}\,x + c_4 \sinh\sqrt{s}\,x}$$

We need to ensure that $\phi(x)$ satisfies the boundary conditions.

$$\phi(0) = 0 \Rightarrow c_3 \cosh\sqrt{s}\,x = 0 \Rightarrow \boxed{c_3 = 0}$$

$$\phi(L) = 0 \Rightarrow c_4 \sinh\sqrt{s}\,L = 0 \Rightarrow \boxed{c_4 = 0}$$

because $\sqrt{s}L > 0$ and $\sinh\sqrt{s}x > 0$ for $x > 0$.

This means that $\phi(x) \equiv 0$ (a trivial solution!) so $\lambda < 0$ is <u>NOT</u> an eigenvalue.

PRODUCT SOLUTIONS and the PRINCIPLE OF Superposition

Summary

We have solved the heat equation $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ satisfying

$u(0,t) = u(L,t) = 0$ corresponding to $\lambda > 0$.

The solutions, $u(x,t) = \phi(x)\,G(t)$, where

$$G(t) = c\,e^{-\lambda k t} \quad \text{and} \quad \phi(x) = c_2 \sin(\sqrt{\lambda}\,x)$$

with $\lambda = \left(\frac{n\pi}{L}\right)^2$, $n \geq 0$.

$$\boxed{u(x,t) = c c_2 \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}, \quad n = 1, 2, 3, \ldots} \qquad (*)$$

are solutions to the heat equation.

$$\lim_{t\to\infty} u(x,t) = 0, \text{ due to the exponential term.}$$

$$u(x,0) = c c_2 \sin\left(\frac{n\pi x}{L}\right)$$

Initial value problems

We can use the form of the product solution $(*)$ to solve an initial value problem provided the initial condition fits a specific form.

e.g

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

$$u(0,t) = 0$$
$$u(L,t) = 0$$
$$u(x,0) = 4\sin\left(\frac{3\pi x}{L}\right)$$

Summary - Eigenvalues.

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 , \quad n = 1, 2, 3, \ldots$$ are called the eigenvalues of and

$$\phi_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$ are the eigenfunctions.

Rationale: $$-\frac{d^2\phi}{dx^2} = \lambda\phi, \qquad \phi(0) = \phi(L) = 0$$

Let $L(\phi)$ denote the operator $-\frac{d^2\phi}{dx^2}$ acting on functions

satisfying $\phi(0) = \phi(L) = 0$.

We can write the ODE as

$$L(\phi) = \lambda\phi \qquad (1)$$

From linear algebra, recall that an eigenfunction is a solution $\phi \neq 0$ of (1).

Remarks

* For an $N \times N$ matrix we have at most N eigenvalue.
* In our case we have an infinite dimensional system wh an infinite number of eigen values

$$\lambda_n = \frac{\pi^2}{L^2}, \frac{4\pi^2}{L^2}, \frac{9\pi^2}{L^2}, \ldots$$

Normal Modes,

In Physics the eigenfunctions are called the normal modes because they are the natural shapes of the solutions that persist for all time.

Our solution from (*) is of the form

$$u(x,t) = B \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}, \quad B = \text{cte a constant}$$

with initial conditions

$$u(x,0) = B \sin\left(\frac{n\pi x}{L}\right).$$

The initial condition of the problem is $4 \sin\left(\frac{3\pi x}{L}\right)$ so

If we pick $B=4$ and $n=3$, then the solution becomes

$$\boxed{u(x,t) = 4 \sin\left(\frac{3\pi x}{L}\right) e^{-k\left(\frac{3\pi}{L}\right)^2 t}}$$

<u>Principle of superposition</u>

If the initial condition takes the form

$$u(x,0) = 4 \sin\left(\frac{3\pi x}{L}\right) + 7 \sin\left(\frac{8\pi x}{L}\right)$$

i.e linear combinations of the general initial condition

$$u(x,0) = B \sin\left(\frac{n\pi x}{L}\right)$$

then the solution is a linear combination of 2 simpler solutions

$$u(x,t) = \underbrace{4 \sin\left(\frac{3\pi x}{L}\right) e^{-k\left(\frac{3\pi}{L}\right)^2 t}}_{B=4,\ n=3} + \underbrace{7 \sin\left(\frac{8\pi x}{L}\right) e^{-k\left(\frac{8\pi}{L}\right)^2 t}}_{B=7,\ n=8}$$

More Superposition.

In general, if $u_1, u_2, \dots, u_M$ are solutions to the linear homogeneous problem, then any linear combination of $\{u_i\}_{i=1}^{M}$ is a solution.

i.e $\quad c_1 u_1 + c_2 u_2 + \dots + c_M u_M = \sum_{n=1}^{M} c_n u_n$ is a solution.

We have established that

$$u(x,t) = \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

is a solution to the linear homogeneous equation then

$$u(x,t) = \sum_{n=1}^{M} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t} \qquad \text{(for any finite } M\text{)}$$

is a solution, provided

$$u(x,0) = f(x) = \sum_{n=1}^{M} B_n \sin\left(\frac{n\pi x}{L}\right)$$

What if the initial condition $u(x,0) = f(x)$ is not a linear combination of sine functions?

FOURIER SERIES!

FACTS

1. Any piecewise continuous function can be approximated by a linear combination of sine functions.

2. As $M \to \infty$, the series converges to $f(x)$. i.e for any initial condition $f(x)$ (piecewise continuous)

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

so that the solution to the heat equation can be written as

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

Construction of Fourier Series

Given an initial condition of the form

$$u(x,0) = f(x)$$

our goal is to write $f(x)$ in the form

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \qquad (*)$$

We need a way to compute B_n.

We introduce the following notion of orthogonality

$$\boxed{\int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0, & m \neq n \\ \frac{L}{2}, & m = n. \end{cases}} \qquad (**)$$

i.e The set of functions $\sin\left(\frac{n\pi x}{L}\right)$ is an orthogonal set.

multiply both sides of $(*)$ by $\sin\left(\frac{m\pi x}{L}\right)$ and integrate

$$\int_0^L f(x) \sin\left(\frac{m\pi x}{L}\right) dx = \sum_{n=1}^{\infty} \int_0^L B_n \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx$$

(here we assume that the series is convergent so we can switch the order of integration & summation)

$$\int_0^L f(x) \sin\left(\frac{m\pi x}{L}\right) dx = B_m \int_0^L \sin^2\left(\frac{m\pi x}{L}\right) dx = \frac{L}{2}$$

because of $(**)$

so $$B_m = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{m\pi x}{L}\right) dx.$$

FACT

For most boundary value problems, the eigenfunctions form an orthogonal set of functions

Example

1D rod with an initial temperature of 100°C and place the ends of the rod in large well-stirred baths of ice water, 0°C.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad t>0, \quad 0<x<L \qquad (P)$$

$$u(0,t)=0, \quad u(L,t)=0, \quad t>0$$

$$u(x,0) = 100.$$

The general solution to (P) is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

$u(x,0) = f(x) = 100°C$

where $\boxed{B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx} = \frac{2}{L}\int_0^L 100\sin\left(\frac{n\pi x}{L}\right)dx.$

$$B_n = \frac{200}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right)dx = \frac{200}{L}\left(-\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\right)\Bigg|_{x=0}^{L}$$

$u = \frac{n\pi x}{L}$

$$= -\frac{200}{n\pi}\cos(n\pi) + \left(\frac{200}{n\pi}\right)$$

$$= \begin{cases} 0, & n \text{ is even} \quad (\cos(n\pi) = 1) \\ \frac{400}{n\pi}, & n \text{ is odd} \quad \cos(n\pi) = -1. \end{cases}$$

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

where $B_n = \begin{cases} 0, & n \text{ even} \\ \frac{400}{n\pi}, & n \text{ odd.} \end{cases}$

* we will come back to the series to reconcile the fact that $u(x,0)=100$ but the series is zero at the boundary points.

Neumann Condition

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \qquad 0<x<1$$

$$\frac{\partial u}{\partial x}(0,t)=0, \qquad \frac{\partial u}{\partial x}(1,t)=0 \qquad \text{(Insulated Rod)}$$

$$u(x,0) = f(x)$$

We can use the method of separation of variables because
(a) PDE is homogeneous i.e $f(x)=0$ / no sources
(b) The boundary conditions are also homogeneous.

Let $u(x,t) = \phi(x)\,G(t)$ so that

$$\frac{dG}{dt} = -\lambda k G \quad \text{and} \quad \frac{d^2\phi}{dx^2} = -\lambda\phi, \quad \lambda \text{ is the separation constant.}$$

recall that this comes from pluggin in $u(x,t) = \phi(x)G(t)$ into the PDE taking the derivatives and separating the variables.

Time dependent problem has solution

$$G(t) = ce^{-\lambda t}.$$

The eigenvalue problem

$$\frac{d^2\phi}{dx^2} = -\lambda\phi$$

$$\frac{d\phi}{dx}(0)=0, \quad \frac{d\phi}{dx}(1)=0.$$

We repeat the eigenvalue analysis as before.

$\underline{\lambda>0}$

$$\phi(x) = c_1\cos(\sqrt{\lambda}x) + c_2\sin(\sqrt{\lambda}x)$$

Applying boundary conditions:

$$\frac{d\phi}{dx} = \Rightarrow \sqrt{\lambda}\left(-c_1\sin\sqrt{\lambda}x + c_2\cos(\sqrt{\lambda}x)\right)$$

$$\frac{d\phi}{dx}(0)=0 \qquad \sqrt{\lambda}\,c_2\cos(\sqrt{\lambda}\cdot 0)=0 \Rightarrow c_2=0 \text{ because } \lambda>0$$

$\frac{d\phi}{dx}(L) = 0 \Rightarrow \sqrt{\lambda}\left(c_1 \sin(\sqrt{\lambda}\, L)\right) = 0 \Rightarrow \sin(\sqrt{\lambda} L) = 0$ as $c_1 = 0$ leads to trivial solution

So we get the same eigenvalues as in the homogeneous dirichlet case

$$\lambda = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, \dots$$

Recall that

$$\phi(x) = c_1 \cos(\sqrt{\lambda}\, x) + c_2 \sin(\sqrt{\lambda}\, x) \quad (\text{and } c_2 = 0)$$

$\Rightarrow$ the eigenfunctions are

$$\phi(x) = c_1 \cos\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, 3, \dots$$

therfore

$$\begin{aligned} u(x,t) &= \phi(x)\, G(t) \\ &= c_1 \cos\left(\frac{n\pi x}{L}\right) c\, e^{-\left(\frac{n\pi}{L}\right)^2 kt}, \quad n = 1, 2, 3, \dots \\ &= A \cos\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt}, \quad n = 1, 2, 3, \dots \quad A = c_1 c \end{aligned}$$

$\underline{\lambda = 0}$

$$\frac{d^2\phi}{dx^2} = 0 \Rightarrow \phi(x) = c_1 + c_2 x$$

$$\boxed{\frac{d\phi}{dx}(0) = 0, \quad \frac{d\phi}{dx}(L) = 0}$$

$$\frac{d\phi}{dx} = c_2$$

$$\frac{d\phi}{dx}(0) = 0 \Rightarrow c_2 = 0$$

$$\frac{d\phi}{dx}(L) = 0 \Rightarrow c_2 = 0$$

This means $\phi(x) = c_1$ is an eigenfunction corresponding to $\lambda = 0$.

Fact $\lambda < 0$ does not yeild any eigenfunctions.

By the principle of superposition,

$$u(x,t) = \underbrace{A_0}_{\boxed{\text{Constant solution corresponding to } \lambda=0}} + \underbrace{\sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt}}_{\boxed{\lambda>0 \text{ solution.}}}$$

$u(x,t)$ can also be written as

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt}$$

The initial condition $u(x,0)$ is satisfied if

$$\boxed{f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)}.$$

Given any function f, we need to write f as a fourier cosine series.

Conveniently $\left\{\cos\left(\frac{n\pi x}{L}\right)\right\}$ is also an orthogonal set of functions due to

$$\int_0^L \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0, & n \neq m \\ \frac{L}{2}, & n = m \neq 0 \\ L, & n = m = 0. \end{cases} \qquad (*)$$

<u>To obtain the FOURIER coefficients</u>

Indeed,

$$\int_0^L f(x) \cos\left(\frac{m\pi x}{L}\right) dx = \sum_{n=0}^{\infty} A_n \int_0^L \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx.$$

using $(*)$ on RHS, we have

$$A_0 = \frac{1}{L}\int_0^L f(x)\, dx \quad \text{and} \quad A_m = \frac{2}{L}\int_0^L f(x) \cos\left(\frac{m\pi x}{L}\right) dx.$$

<u>Note</u> $\lambda = 0$

$$u(x,t) = A_0 = \frac{1}{L}\int_0^L f(x)\, dx$$

$\lambda > 0$.

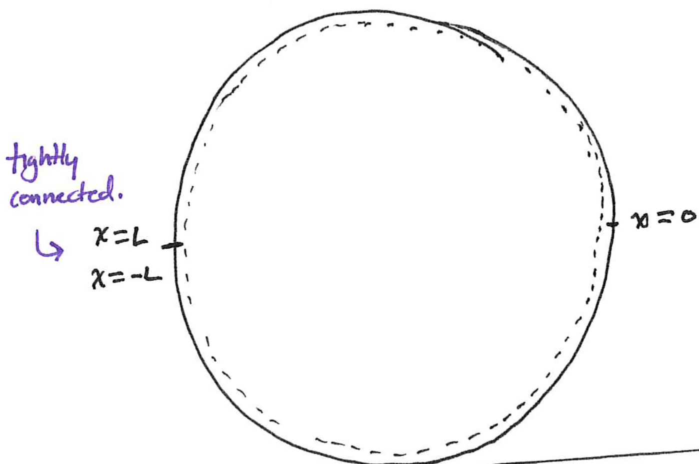
$$\sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt} \to 0 \quad \text{as } t \to \infty$$

therefore

$$\lim_{t\to\infty} u(x,t) = A_0 = \frac{1}{L}\int_0^L f(x)\,dx$$ (the solution to the steady state problem).

HEAT CONDUCTION IN A THIN CIRCULAR RING

(Eigenfunctions that are combinations of sine and cosine functions)



PDE

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad -L < x < L, \quad t > 0$$

$u(-L,t) = u(L,t)$ (perfect thermal contact).

$\frac{\partial u}{\partial x}(-L,t) = \frac{\partial u}{\partial x}(L,t)$ (continuous heat flux)

$u(x,0) = f(x)$

Boundary conditions of this type are called periodic $-L < x < L$

First, we note that we have a linear homogeneous PDE with linear boundary conditions so we can use the method of separation of variables.

Let $u(x,t) = \phi(x)\,G(t)$

As before, $G(t) = ce^{-\lambda k t}$, with the eigenvalue problem

$$\frac{d^2\phi}{dx^2} = -\lambda\phi$$

(EVP) $\phi(-L) = \phi(L)$

$$\frac{d\phi}{dx}(-L) = \frac{d\phi}{dx}(L)$$

B.c of this type are called MIXED / ~~[illegible]~~ boundary conditions.

The general solution to (EVP) is

$$\phi(x) = c_1 \cos(\sqrt{\lambda}\, x) + c_2 \sin(\sqrt{\lambda}\, x)$$

then applying boundary conditions yeilds

$$\phi(-L) = \phi(L) \Rightarrow \boxed{\begin{aligned} &c_1 \cos(\sqrt{\lambda}(-L)) + c_2 \sin(\sqrt{\lambda}(-L)) \\ &= c_1 \cos(\sqrt{\lambda}\, L) + c_2 \sin(\sqrt{\lambda}\, L) \end{aligned}}$$

As cosine is even

$$\cos(\sqrt{\lambda}\, L) = \cos(\sqrt{\lambda}(-L)) \qquad (*)$$

and sine is odd

$$\sin(\lambda(-L)) = -\sin(\sqrt{\lambda} L) \qquad (**)$$

we have $\boxed{2\sin(\sqrt{\lambda}\, L)\cdot c_2 = 0} \Rightarrow \boxed{c_2 \sin(\sqrt{\lambda}\, L) = 0}$ (1)

The second boundary conditions

$$\frac{d\phi}{dx}(-L) = \frac{d\phi}{dx}(L) \qquad \left(\frac{d\phi}{dx} = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}x) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}x) \right)$$

$$\Rightarrow -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}(-L)) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}(-L)) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}L) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}L)$$

Using (*) and (**), we have

$$-2c_1 \sin(\sqrt{\lambda}\, L) = 0 \Rightarrow \boxed{c_1 \sin(\sqrt{\lambda}\, L) = 0} \qquad (2)$$

If $\sin\sqrt{\lambda} L \neq 0$, then both c_1 and c_2 are zero in (1) and (2) yeilding the trivial solution.

It must be the case

$$\sin(\sqrt{\lambda}\, L) = 0 \Rightarrow \sqrt{\lambda} L = n\pi, \quad n = 1, 2, 3, \ldots$$

so we have the same eigenvalues as before $\lambda = \left(\frac{n\pi}{L}\right)^2$, and c_1 and c_2 are arbitrary!

where the initial condition is satisfied if

$$u(x,0) = f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

therefore given $f(x)$ we find a_0, a_n, b_n.

$$f(x) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

using the following orthorgonality conditions

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0, & n \neq m \\ L, & n = m \neq 0 \\ 2L, & n = m = 0 \end{cases},$$

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0, & n \neq m \\ L, & n = m \neq 0, \end{cases}$$

and

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx = 0$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\, dx \qquad a_m = \frac{1}{L}\int_{-L}^{L} f(x) \cos\left(\frac{m\pi x}{L}\right) dx$$

$$b_m = \frac{1}{L}\int_{-L}^{L} f(x) \sin\left(\frac{m\pi x}{L}\right) dx$$

this means

$$\phi(x) = C_1 \cos\left(\frac{n\pi x}{L}\right) + C_2 \sin\left(\frac{n\pi x}{L}\right)$$

is an eigenfunction.

Infact both $\cos\left(\frac{n\pi x}{L}\right)$ and $\sin\left(\frac{n\pi x}{L}\right)$ are eigenfunctions

$n = 1, 2, 3, \ldots$

corresponding to $\lambda = \left(\frac{n\pi}{L}\right)^2$.

This yields solutions

$$u(x,t) = \cos\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt} \quad \& \quad u(x,t) = \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt}$$

corresponding to $\underline{\lambda > 0}$

$\underline{\lambda = 0}$

$$\phi(x) = C_1 + C_2 x$$

Applying B.C yields

$$\phi(-L) = \phi(L) \Rightarrow C_1 - C_2 L = C_1 + C_2 L \Rightarrow 2C_2 L = 0 \Rightarrow C_2 = 0.$$

$$\frac{d\phi}{dx} = C_2 = 0$$ so C_1 is arbitrary

$$\phi(x) = C_1$$ (Any constant function is an eigen function)

$\lambda < 0$, has no eigenfunctions

By the principle of super-position of all product solutions

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt} + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt}$$

$\uparrow$ $\lambda = 0$ $\qquad$ $\uparrow$ $\lambda > 0$