WAVE Equation

GOAL : Describe vibrations of a perfectly elastic string.

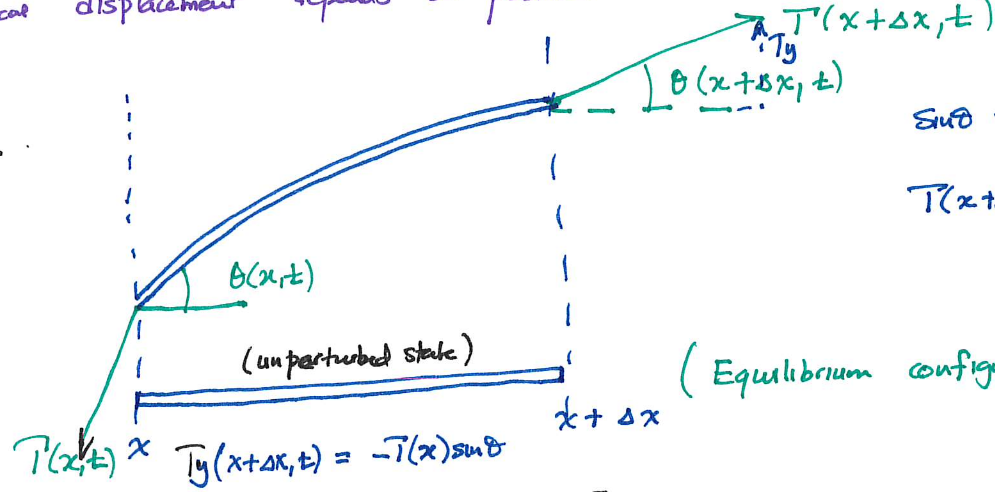
Assumptions (Traking the motion of each particle that makes up the string)

1. The slope of the string is small $\Rightarrow$ horizontal displacement is negligable.
2. The string is tightly stretched (think musical instrument)
3. The string is perfectly flexible (no resistance to bending)

We want to keep track of the vertical displacement, $y = u(x,t)$

i.e vertical displacement depends on position (x) and time, t.

Newton's Law



$Ty(x+\Delta x, t) = -T(x)\sin\theta$

$\sin\theta = \dfrac{Ty(x+\Delta x,t)}{T(x+\Delta x,t)}$

$T(x+\Delta x,t)\sin\theta = Ty(x+\Delta x$

Consider an infinitesimally thin segment $[x, x+\Delta x]$

Let $\rho_0(x)$ be the mass-density, then

$$\text{Total mass} = \rho_0(x)\,V \cong \rho_0(x)\,\Delta x.$$

FORCES

1. Body forces - acting in the vertical direction (e.g gravitational force)
2. forces acting on the ends of the segment
3. Tension in the string (force exerted by the rest of the string on the endpoints of the segments of the string is in the tangental direction)

Let θ be the angle between the horizontal and the string

$$\text{Slope of string} = \frac{dy}{dx} \quad \text{or} \quad \tan\theta(x,t) = \frac{\Delta y}{\Delta x} \qquad \text{since } y = u(x,t)$$

$$\text{Slope} = \frac{\partial u}{\partial x}$$

Neglecting horizontal motion

$$F = ma$$

$$\underbrace{\rho_0(x)\,\Delta x}_{\text{mass}}\ \underbrace{\frac{\partial^2 u}{\partial t^2}}_{\text{accelaration}} = \underbrace{T(x+\Delta x,t)\sin\theta(x+\Delta x,t) - T(x,t)\sin\theta(x,t)}_{\text{vertical components of tension}}$$

$$+ \underbrace{\rho_0(x)\,\Delta x\, Q(x,t)}_{\text{Body force.}}$$

where $Q(x,t)$ is the body force per unit mass.

dividing by Δx

$$\rho_0(x)\frac{\partial^2 u}{\partial t^2} = \frac{T(x+\Delta x,t)\sin\theta(x+\Delta x,t) - T(x,t)\sin\theta(x,t)}{\Delta x} + \rho_0(x)\,Q(x,t)$$

taking $\lim \Delta x \to 0$

$$\rho_0(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left[T(x,t)\sin\theta(x,t)\right] + \rho_0(x)\,Q(x,t)$$

For small angles, θ $\quad \frac{\partial u}{\partial x} = \tan\theta = \frac{\sin\theta}{\cos\theta} \cong \sin\theta$

so

$$\boxed{\rho_0(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left[T\frac{\partial u}{\partial x}\right] + \rho_0(x)\,Q(x,t)}$$

Perfectly Elastic Strings

$T(x,t) \rightsquigarrow$ depends on local stretching of the string, thus is assumed to be the same for the unperturbed horizontal stretching string as well as the stretchy string because θ is small.

$$T(x,t) \cong T_0 \text{ (constant)}.$$

$$\rho_0(x)\frac{\partial^2 u}{\partial t^2} = T_0\frac{\partial^2 u}{\partial x^2} + Q(x,t)\,\rho_0(x)$$

1D wave equation

If the body force per unit mass is gravitational, i.e. $Q(x,t) = g$

$$\rho_0 g \lll \left| T_0 \frac{\partial^2 u}{\partial x^2} \right|$$ so we can

neglect $Q(x,t)\,\rho_0(x)$ so that

$$\boxed{\rho_0(x)\frac{\partial^2 u}{\partial t^2} = T_0\frac{\partial^2 u}{\partial x^2}}$$

$$\Downarrow$$

1D wave Equation.

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}}$$

where $c^2 = \dfrac{T_0}{\rho_0(x)}$

This notation is introduced because $\dfrac{T_0}{\rho_0(x)}$ has units m^2/s^2

so c is a velocity.

Boundary conditions

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

We have 2 spatial derivatives so we can apply one b.c on each end.

e.g <u>FIXED STRING</u>

$$u(L,t) = 0.$$

Variable string

$$u(L,t) = f(t)$$

Vibrating string with fixed ends

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad (P)$$

B.C $\quad u(0,t) = 0$
$\quad\quad\;\; u(L,t) = 0$

Initial conditions * $\quad u(x,0) = f(x) \quad$ (initial position)
$\quad\quad \frac{\partial u}{\partial t}(x,0) = g(x) \quad$ (initial velocity)

* Since we have 2 time derivatives, we need to specify 2 conditions at $t=0$.

Both the Pde and boundary conditions are homogeneous so we can apply the method of separation of variables!

Indeed, let $u(x,t) = \phi(x)\,h(t)$, plug into (P)

$$\phi(x)\frac{d^2h}{dt^2} = c^2 h(t)\frac{d^2\phi}{dx^2}$$

divide on both sides by $c^2\phi(x)\,h(t)$,

$$\frac{1}{c^2}\frac{1}{h}\frac{d^2h}{dt^2} = \frac{1}{\phi}\frac{d^2\phi}{dx^2} = -\lambda,$$ where λ is the separation constant.

$$\frac{d^2h}{dt^2} = -\lambda c^2 h$$

$$\boxed{\begin{array}{l}\dfrac{d^2\phi}{dx^2} = -\lambda\phi. \\[1em] \phi(0) = \phi(L) = 0\end{array}} \quad \text{BVP}$$

If (i) $\lambda > 0$,
$$\frac{d^2h}{dt^2} = -\lambda c^2 h$$
$$h(t) = c_1 \cos(c\sqrt{\lambda}\, t) + c_2 \sin(c\sqrt{\lambda}\, t)$$

(ii) $\lambda = 0$
$$h(t) = c_1 + c_2 t$$

$c_1 e^{c\sqrt{\lambda}t} + c_2 e^{-c\sqrt{\lambda}t}$

linear comb of $\cosh z$ & $\sinh z$.

(iii) $\lambda < 0$, $h(t)$ is exponentially growing and decaying in time.

Since we are solving the wave equation $\lambda > 0$ is reasonable.

The BVP, has eigenvalues $\lambda = \left(\frac{n\pi}{L}\right)^2$ $n = 1, 2, 3, \ldots$

with eigenfunctions $\sin\left(\frac{n\pi x}{L}\right)$ therefore by the principle of superposition

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi c t}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi c t}{L}\right)$$

The initial conditions are satisfied if

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$$

$$\frac{\partial u}{\partial t}(x,t) = -\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)\left(\frac{n\pi c}{L}\right)\sin\left(\frac{n\pi c t}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)\left(\frac{n\pi c}{L}\right)\cos\left(\frac{n\pi c t}{L}\right)$$

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} B_n \left(\frac{n\pi c}{L}\right)\sin\left(\frac{n\pi x}{L}\right) = g(x)$$

From the orthogonality of $\left\{\sin\left(\frac{n\pi x}{L}\right)\right\}$ we can compute the coefficients

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$B_n\left(\frac{n\pi c}{L}\right) = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi c t}{L}\right) = \underbrace{\frac{1}{2}\cos\left(\frac{n\pi}{L}(x-ct)\right)}_{\text{wave travelling to the right } \bar{w} \text{ velocity } c} - \underbrace{\frac{1}{2}\cos\left(\frac{n\pi}{L}(x+ct)\right)}_{\text{wave travelling to the left with velocity } c}$$

$$u(x,t) = R(x-ct) + S(x+ct)$$

* $\sin\alpha \sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$
* $\sin\alpha \cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$

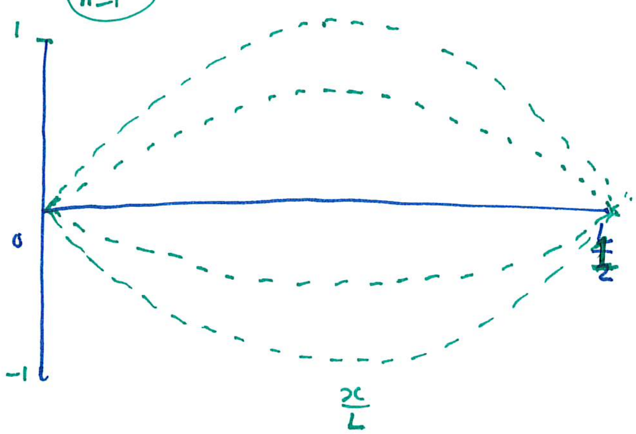
Desmos

* $f(x) = \cos(\pi(x-t))$

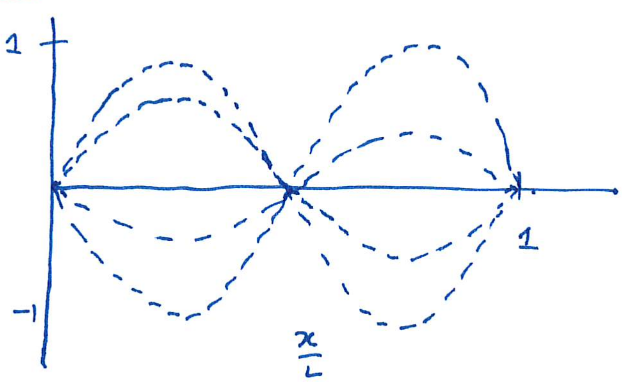
travelling wave moving to the right

vibrating string - normal modes.

n=1



n=2



Interpretation of solution

The vertical displacement is composed of a linear combination of

$$\sin\left(\frac{n\pi x}{L}\right)\left(A_n \cos\left(\frac{n\pi c t}{L}\right) + B_n \sin\left(\frac{n\pi c t}{L}\right)\right)$$

These are called the NORMAL MODES of vibration.

- Intensity of sound depends on amplitude $\sqrt{A_n^2 + B_n^2}$
- Time dependence is simple harmonic with a circular frequency (# of oscillations in 2π units of time)

$$\text{\# of oscillations} / 2\pi \text{ units of time} = \frac{n\pi c}{L} \quad (\text{Natural frequency})$$

where $c = \sqrt{\frac{T_0}{\rho_0}}$.

- The sound produced is a super-position of the infinite number of natural frequencies $(n = 1, 2, 3, \dots)$ (Natural frequencies are $c\sqrt{\lambda} = \frac{n\pi c}{L}$, $\lambda = \left(\frac{n\pi}{L}\right)^2$)

$n=1$ (First harmonic / Fundamental)

circular frequency of $\frac{\pi c}{L}$.

The larger the natural frequency, the larger the pitch of sound produced.

To produce a desired fundamental frequency c or L can be varied.

$c = \sqrt{\frac{T_0}{\rho_0}}$ so the Tension can be varied

Standing waves

At each t, each mode looks like a simple oscillation in x.

This is called a standing wave.

Vibrating membrane

Vertical displacement of a vibrating string is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

2D wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 y}{\partial y^2} \right)$$

is the vibration of a highly stretched membrane

(We will come back to solve this)